HYDROSTATICS

for the B.A. & B.Sc. Courses

BY

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PREFACE

This book has been written to satisfy the long-felt need for a text-book on Hydrostatics suited to the requirements of the South Indian Universities. The topics dealt with in the author's Manual of Physics for the Intermediate course, Vol. I, have generally been omitted, and the student is advised to go through the section on Hydrostatics in that book for a preliminary study, if he has not already done so. Further, the topics which are decidedly outside the syllabuses of the South Indian Universities have also been omitted.

The scope of this book comprises the syllabuses of the Madras, Andhra, Annamalai, Travancore and Mysore Universities. The Madras University B.Sc. students may omit the following articles :— 14, 19.

The Madras University B. A. students may omit the following articles :—

12, 14, 19, 21, 22, 43, 58, 60, 66, 67, 68, 80, 81.

Corrections, criticisms and suggestions for improvement will be thankfully received by the author.

Thyagarayanagar,
July 1940.

T. V. VENKATACHARI.

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CHAPTER I

SIMPLE THEOREMS RELATING TO PRESSURE

1. Solids, Liquids and Gases.— A *solid* is a body which offers *permanent* resistance to small forces tending to alter its shape, *i.e.*, it possesses *rigidity*. A *fluid* (liquid or gas) is a body which offers *no permanent* resistance to forces tending to alter its shape, *i.e.* it has no rigidity.

All bodies (solid or fluid) possess *volume elasticity*. Of fluids some possess a high degree of volume elasticity while others possess very little. The former are called *liquids* and the latter *gases*. *Liquids* are fluids which offer *very great* resistance to forces tending to diminish their volume, and which, when placed in an empty vessel present a *free surface*. Gases are fluids which offer *only a small resistance* to forces tending to diminish their volume, and which, when introduced into an empty vessel, fill it completely and have consequently no free surface.

2. Normal Force and Shearing Force.— A force applied to the surface of a body so as to be perpendicular to the surface is called a *normal* force; and a force applied to the surface so as to be parallel to the surface is called a *tangential* or *shearing* force. The normal force will tend to compress the body if it is a push or to extend the body if it is a pull; the former is called a **thrust** and the latter a **tension** (Fig. 1).

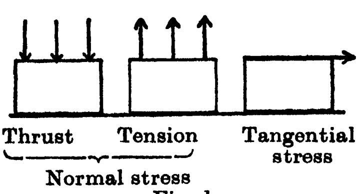

Fig. 1.

3. Thrust and Pressure.—Thrust, as we have seen, is the normal force applied towards the surface of a body so as to tend to compress it. The **pressure** on a surface is the normal force towards the surface (or the thrust) *per unit area.*

The thrust on a surface may be uniformly distributed, or not. In the former case the thrust will be the same on *every* equal area of the surface. When a thrust is uniformly distributed over a surface, the thrust on unit area gives the pressure on the surface. For example, if a force F is uniformly distributed as a thrust over a surface of area A, the pressure P on the surface is given by

$$P = F/A.$$

If the thrust on a surface is not uniformly distributed, the pressure on the surface will be different at different parts. The **pressure at a point** on the surface is then defined as the ratio of the thrust on a small portion of the surface including the point to the area of that portion, when the area becomes indefinitely small. For example, let the pressure at the point P (Fig. 2) be required. Take a small surface having P at about the middle. Let a be the area of the surface and f the thrust on it. Then $f\,/\,a$ gives the *mean pressure* on the surface. If we now make the surface smaller and smaller until it practically coincides with the point P, the value of f as well as a will become smaller and smaller, and the ratio f/a in the limit when the surface practically coincides with P gives the pressure at the point P.

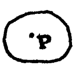

Fig. 2.

4. Force on surface of Fluid at Rest always Normal.— Since a fluid offers no permanent resistance to

continued tangential or shearing force, however small, the force on any surface AB (Fig. 3) in a liquid *at rest* must be entirely normal to that surface. For, if there were a force acting on AB in any other direction, there would be a component of that force tangential to AB to which the fluid will yield, resulting in motion.

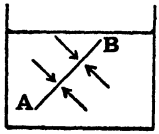

Fig. 3

5. **Density and Specific Gravity.**—The **density** of a substance is its mass per unit volume. It is expressed in grams per c.c. or lb. per c. ft. If M gm. of a substance occupies a volume of V c.c., then the density d of the substance is given by

$$d = M/V \text{ gm. per c.c.}$$

Also $$M = Vd \text{ gm. and}$$

$$V = M/d \text{ c.c.}$$

The **specific gravity** of a substance is the *ratio* of the mass of any volume of the substance to the mass of an equal volume of pure water at 4°C. It is the same as the ratio of the density of the substance to the density of pure water at 4°C. Specific gravity is expressed as a mere number.

The specific gravity of a substance is the same in the C. G. S. and F. P. S. systems, since the *ratio* between the masses of two bodies must be the same in any system of units.

The density of a substance in the C. G. S. system is *numerically* the same as its specific gravity, since the mass of 1 c.c. of pure water at 4°C. is 1 gm. But in the F. P. S. system, since the mass of 1 c. ft. of pure water at 4°C. is 62·4 lb., the density of a substance is numerically equal to 62·4 times its specific gravity.

6. The Pressure at a Point in a Fluid at rest is the Same in All Directions.—Take any point C in the fluid of density ρ. Consider the equilibrium of a

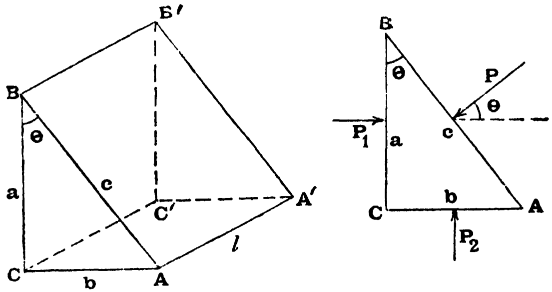

Fig. 4.

small right $\triangle$r prism of the fluid CABC′A′B′ (Fig. 4), such that CA and CC′ are horizontal and at right angles to each other and CB vertical. Let CB $= a$, CA $= b$, AB $= c$ and CC′ $= l$, and let $\angle$CBA $= \theta$. Let P_1 and P_2 be the pressures in the directions CA and CB respectively and P the pressure at right angles to ABB′A′. The dimensions of the prism are taken to be so small that the pressure is uniform on each face.

The prism is in equilibrium under the action of the following forces.—

1. Thrust on BCC′B′ $= P_1\, al$, parallel to CA.

2. Thrust on CAA′C′ $= P_2\, bl$, parallel to CB.

3. Thrust on ABB′A′ $= P\, cl$, at angle θ with the horizontal.

4. The weight of the liquid prism, $\frac{1}{2}\, abl\, \rho g$ acting vertically down.

5 and 6. Thrusts on the ends BCA and B′C′A′, parallel to the line CC′

Resolving the forces horizontally parallel to CA, we have

$$P_1\, a\, l = P\, c\, l \cos\theta = P\, a\, l$$

$$\therefore\ P_1 = P.$$

Resolving vertically, we have

$$P_2\, b\, l = P\, c\, l\ \sin\theta + \tfrac{1}{2}\, a\, b\, l\, \rho g.$$

$$i.\ e.,\quad P_2\, b\, l = P\, b\, l + \tfrac{1}{2}\, a\, b\, l\, \rho g,$$

$$i.\ e.,\quad P_2 = P + \tfrac{1}{2}\, a\rho g.$$

If the prism becomes so small as to approximate to the point C itself, the quantity $\frac{1}{2}\, a\, \rho g$ becomes negligibly small (on account of the factor a) in comparison with P. Hence

$$P_2 = P$$

Thus P_1, P_2 and P are all equal at the point C. Here P may act at any angle θ with the horizontal and in any vertical plane we choose, and hence *the pressure at any point* C *is the same in all directions.*

From the above proof we see that when a body is broken into very small pieces, the weight, depending upon the volume, becomes insignificant in comparison with any surface force, such as thrust or viscous resistance, depending upon the area. If a piece of stone is dropped from above, it falls down rapidly. But if it is ground into very fine powder and strewn about, the particles float in air as dust and take a very long time to settle down. Again, small particles of water float above as cloud and come down only when they aggregate to form comparatively bigger particles of rain.

For the sake of simplicity let us suppose that in the case of the prism considered above b and l are each equal to a cm., that the density of the liquid is equal to 1 gm. per c.c. and that the pressure P_2 is equal to 1 gm. wt. per sq. cm. The volume of the prism is $\frac{1}{2}\, a^3$ c.c. and its weight $\frac{1}{2}\, a^3$ gm. wt. The area of the face CAA′C′ is a^2 sq. cm. and the

thrust on it is a^2 gm. wt. Let us compare the weight with the thrust as a goes on decreasing :

Value of a	Weight of prism	Thrust on face	$\frac{\text{Weight}}{\text{Thrust}}$
1 cm.	$\frac{1}{2}$ gm. wt.	1 gm. wt.	$\frac{1}{2}$
$\frac{1}{10}$ cm.	$\frac{1}{2\times10^3}$,,	$\frac{1}{10^2}$,,	$\frac{1}{20}$
$\frac{1}{100}$ cm.	$\frac{1}{2\times10^6}$,,	$\frac{1}{10^4}$,,	$\frac{1}{200}$
...	...	...	...

Thus as a becomes smaller and smaller, the weight becomes less and less significant in comparison with the thrust.

7. In a Fluid at Rest the Pressure is the same at All Points in a Horizontal Plane :—Let A and B (Fig. 5) be any two such points. Join AB. About AB as axis describe a very thin cylinder of sectional area a. Let the

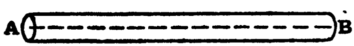

Fig. 5

pressures at A and B be P_1 and P_2 respectively. The cylinder is in equilibrium under the action of the following forces :—

(1) Its weight acting in a direction perpendicular to AB.

(2) The thrusts on the curved surface, which are everywhere perpendicular to the surface and therefore to the axis AB.

(3) The thrusts on the ends, P_1a at A and P_2a at B.

Resolving the forces parallel to AB, we get

$$P_1 a = P_2 a.$$

$$\therefore \quad P_1 = P_2.$$

8. Difference between the Pressures at two Points in a Homogeneous Fluid at Rest: —Let A and B be the two points, A being vertically above B (Fig. 6). Let AB $= h$. About AB as axis describe a very thin cylinder of sectional area a. Let the pressures at A and B be P_1 and P_2 respectively, and let ρ be the density of the fluid. The cylinder is in equilibrium under the action of the following forces:—

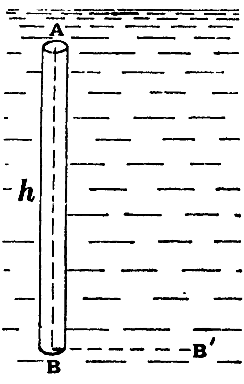

Fig 6

(1) The thrusts on the curved surface, everywhere normal, hence horizontal.

(2) Its weight equal to $ha\rho g$, acting vertically down.

(3) The thrust on the end A equal to $P_1 a$ acting vertically down.

(4) The thrust on the end B equal to $P_2 a$ acting vertically up.

Resolving the forces vertically, we get for equilibrium,

$$P_2 a - P_1 a - ha\rho g = 0$$

$$\therefore \quad P_2 - P_1 = h\rho g \quad .$$

If the lower point (B′) is not vertically below A, draw the vertical line through A and the horizontal line through B′ to meet at B. By Art. 7 the pressure at B′ = the pressure at B. Therefore the difference between the pressures

at A and B′ $= h\rho g$, where h is the difference in level between A and B′.

Corollaries :—(1) The pressure due to a liquid of density ρ at depth h below the surface is given by $h\rho g$.

(2) Since the difference between the pressures at any two given points in a liquid at rest is always equal to $h\rho g$, where h is the difference of level between the points, an increase of pressure at one point must produce an equal increase of pressure at the other point. This is **Pascal's law**, stated as follows :—

An increase of pressure at any point in a liquid at rest is transmitted without change to every other point in the liquid.

For further explanation of this law and its application to the Bramah press, see the author's 'A Manual of Physics,' Vol. I.

9. The surface of a Liquid at Rest is Horizontal.—Let AB (Fig. 7) represent a liquid surface exposed to a uniform pressure p. If AB were not horizontal, draw the horizontal surface AC, and from B draw BC the vertical plane. Taking any right vertical section, A B C, the pressure at A = the pressure at B.

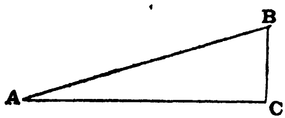

Fig. 7

But the pressure at A = the pressure at C (Art. 7).

$\therefore$ the pressure at B = the pressure at C.

But the pressure at C − the pressure at B = BC ρg (Art. 8).

$$\therefore \text{BC}\,\rho g = 0;\ \text{hence BC} = 0$$

Therefore, AB must be horizontal.

10. The Common Surface of two Liquids which do not mix is a Horizontal Plane.—Let ρ_1 and ρ_2 be the densities of the two liquids ($\rho_2 > \rho_1$). Take two points P and Q (Fig. 8) in the lower liquid in the same horizontal line. Draw PAA′ and QBB′ vertically to meet the common surface in A and B and the surface of the upper liquid in A′ and B′.

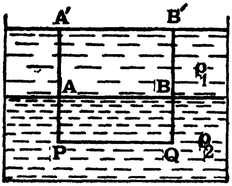

Fig. 8.

Since PQ is horizontal,

the pressure at P = the pressure at Q (Art. 7).

Also, the pressure at A′ = the pressure at B′.

$\therefore\ \rho_1\,AA' + \rho_2 . PA = \rho_1\,BB' + \rho_2 . QB$ (omitting the common g).

i.e., $\rho_1\ AA' + \rho_2\ (PA' - AA') =$

$$\rho_1\ BB' + \rho_2\ (QB' - BB').$$

Since PQ and A′ B′ are horizontal (Art. 9),

$$PA' = QB'$$

$$\therefore\ (\rho_1 - \rho_2)\,AA' = (\rho_1 - \rho_2)\,BB'.$$

$$\therefore\qquad AA' = BB'.$$

$\therefore$ A B is parallel to A′B′ and is therefore horizontal. That is, the straight line joining any two points on the common surface is horizontal. Hence the common surface is horizontal.

11. Thrust on a Plane Surface :—Let a *plane* surface be exposed to a homogeneous liquid (at rest) of density ρ. We shall now calculate the thrust exerted by the liquid (alone) on the surface. Let S be the area of the surface. Divide it up into very small portions or elements. Let the area of one such element be a and its depth below the liquid surface h. The thrust exerted on this element is $h\rho g . a$ (*i.e.*, pressure $\times$ area), and is normal to the surface.

Since the surface is plane, the thrusts on all the elements are parallel and in the same direction, and the resultant thrust is therefore got by adding the thrusts on the various elements. Hence

resultant thrust $= \sum h \rho g a = \rho g \sum a h.$

Let h_1 be the depth of the centre of gravity (or centre of surface) of the plane area. Then from definition of C. G.

$$h_1 \sum a = \sum ah.$$

Hence resultant thrust, $\rho g \sum a h = \rho g h_1 \sum a = h_1 \rho g S.$

Now $h_1 \rho g$ is the pressure due to the liquid at the centre of gravity. Hence *the thrust on any plane surface exposed to a homogeneous liquid is equal to the pressure at the centre of gravity of the surface multiplied by the area of the surface.*

The weight of a column of liquid standing on this area S and having a *uniform* depth h_1 is obviously equal to this thrust. Hence *the thrust on any plane area exposed to a homogeneous liquid is equal to the weight of a column of the liquid whose base is equal to the area and whose (uniform) height is equal to the depth of the centre of gravity of the area below the surface of the liquid.*

12. Resultant Vertical Thrust on any Surface: – Let PRQS (Fig. 9) be *any* surface (plane or curved) immersed in a liquid. Through each point of the bounding edge of this surface imagine a vertical line to be drawn, and let the points in which these vertical lines meet the surface of the liquid form the closed curve ACBD.

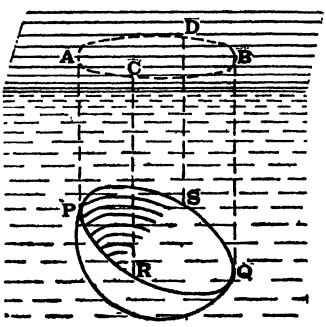

Fig. 9.

Take any element O on the curved surface (Fig. 10). The thrust on this, represented by MO, is normal to the surface at O. This can be resolved into a horizontal and a vertical component, represented by M N and NO respectively. The sum of the vertical components for all the elements constituting the surface is called the **resultant vertical thrust.**

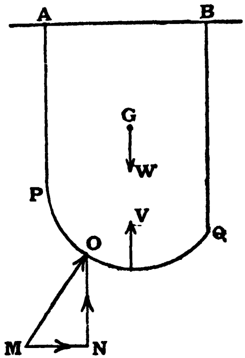

Fig. 10.

Considering the equilibrium of the vertical cylinder of liquid standing upon each element of surface (cf. Art.8), and resolving the forces vertically, we see that the vertical component of the thrust on the element must balance the weight of the superincumbent cylinder of liquid. Therefore the resultant of all these elementary vertical thrusts, *i.e.*, the resultant vertical thrust must be equal and opposite to, and in the same line of action as, the resultant of the weights of these elementary cylindrical columns. But this latter resultant is equal to the weight of the vertical column of liquid standing upon the whole surface and acts at its centre of gravity. Also the thrust due to the liquid upon the surface is equal and opposite to the thrust exerted at the surface upon the liquid. *Hence the resultant vertical thrust on any surface (plane or curved) immersed in a liquid is equal to the weight of the superincumbent liquid and acts through the centre of gravity of this liquid column.*

[In all the above cases, only the thrust due to the liquid has been considered, and this is exclusive of the thrust due to the atmosphere.]

Consider the surface S in a flask filled with a liquid (Fig. 11). If the column of liquid LS were present, the weight of this column would be balanced by the resultant vertical thrust on S. As it is, however, this resultant vertical thrust, equal to the weight of the column, is exerted upwards upon the part S of the vessel, which is thus strained to that extent. It is the reaction due to this strain that balances the resultant vertical thrust. The strain here is not, of course, entirely vertical but we consider here only its vertical component

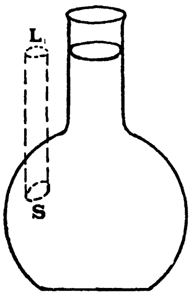

Fig. 11.

Now consider a conical vessel ABC (Fig. 12) filled with a liquid of density ρ. Let the area of the base be a and the altitude h. Then the pressure on the bottom is $h\rho g$ and the thrust on the bottom is $h\rho ga$ (due to the liquid). But the weight of the liquid is only $\frac{1}{3}\, ah\rho g$ ($\because$ vol. of cone $= \frac{1}{3}\, ah$). How can a liquid exert a downward thrust three times as great as its own weight?

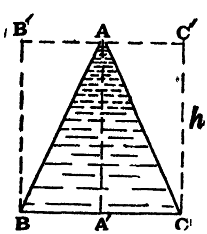

Fig. 12.

The volume of the cylinder BCC'B' is ah. Subtracting the volume of the cone $\frac{1}{3}\, ah$, the remaining volume is $\frac{2}{3}\, ah$. This is the volume of the *virtual* superincumbent liquid on the sides of the vessels. Therefore, the resultant vertical thrust upwards on the sides of the vessel is equal to $\frac{2}{3}\, ah\rho g$. Owing to the reaction of the sides, there is a downward vertical thrust on the liquid equal to this. Thus the thrust on the bottom is the same as if there were a cylinder of liquid BB'C'C above it.

If the conical vessel, emptied, is placed in the pan of a balance and counterpoised, and the vessel then filled with the liquid, what additional weight in the other pan will balance the liquid ? It must, of course, be equal to the weight of the liquid added. But the thrust on the bottom is equal to the weight of the cylinder of liquid BB′C′C.

While the bottom is thrust *down* with a force equal to the weight of the cylinder of the liquid, the sides are thrust *up* with a force equal to the weight of the *virtual* superincumbent liquid. Hence the resultant force due to the liquid on the whole vessel is equal to the weight of the liquid actually present in the vessel.

The resultant of the horizontal components of the thrusts on the sides of the vessel is obviously zero.

EXAMPLES

1. A rectangular area is immersed in water with its plane vertical and a side in the surface. Show how to divide the surface into two parts by a horizontal line so that the thrusts on the two parts may be equal.

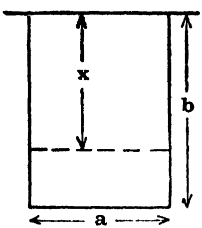

Fig. 13.

Thrust on upper part (Fig. 13) = $\frac{1}{2}\, x \rho g . ax$. (Thrust on plane surface = pressure at C.G. × area.)

Thrust on lower part =

$$\{ x + (b-x)/2 \} \ \rho g \times a\,(b-x)$$

Equating the two thrusts, $x = b / \sqrt{2}$.

2. A rectangle is immersed in water vertically with one side in the surface. Show how to divide it into n parts by horizontal lines so that the thrusts on these parts are equal. Show that the depths of the dividing lines are proportional to the square roots of the natural numbers.

Let a be the length of the horizontal side and b the altitude of the rectangle.

Let the horizontal lines be at depths, x_1, x_2, x_3, etc., below the surface of the liquid.

The thrust on the first (topmost) part $= \frac{1}{2} x_1 \rho g \times a x_1$
$= \frac{1}{2} a x_1^2 \rho g.$

The thrust on the whole rectangle $= \frac{1}{2} b \rho g \times a b$
$= \frac{1}{2} a b^2 \rho g.$

By problem, $n \times \frac{1}{2} a x_1^2 \rho g = \frac{1}{2} a b^2 \rho g.$ Hence

$$x_1 = b / \sqrt{n}.$$

Thrust on first and second parts together $= \frac{1}{2} x_2 \rho g \, a x_2 = \frac{1}{2} a x_2^2 \rho g.$ This must be twice the thrust on the first part.

$\therefore \ \frac{1}{2} a x_2^2 \rho g = 2 \times \frac{1}{2} a x_1^2 \rho g.$ Hence

$$x_2 = x_1 \sqrt{2} = b \sqrt{2} / \sqrt{n}.$$

Thrust on first r parts $= \frac{1}{2} x_r \rho g \, a x_r = \frac{1}{2} a x_r^2 \rho g = r$ times the thrust on first part $= r . \frac{1}{2} a x_1^2 \rho g.$ Hence

$$x_r = x_1 \sqrt{r} = b \sqrt{r} / \sqrt{n}.$$

3. A triangle immersed vertically in a liquid has its base horizontal and vertex in the surface. Divide it by a horizontal line into two parts so that the thrusts on them are equal.

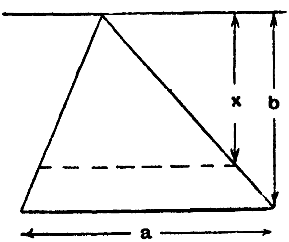

Fig. 14.

Let the horizontal line be at depth x below the liquid surface (Fig. 14).

Thrust on upper part $=$
$\frac{2}{3} x \rho g \times \frac{1}{2} x \, a x / b = \frac{1}{3} x^3 \rho g a / b.$

Thrust on whole triangle $=$
$\frac{2}{3} b \rho g \times \frac{1}{2} a b = \frac{1}{3} a b^2 \rho g.$

By problem, $\frac{1}{3} a b^2 \rho g =$
$2 \times \frac{1}{3} x^3 \rho g a / b.$

Hence $x = b / \sqrt[3]{2}.$

4. On one side of a rectangular vertical gate there is salt water to a depth of 25 ft. and on the other side there is fresh water. Find its depth if the thrusts on the two sides are equal. Sp. gr. of salt water $= 1.025$.

Let a be the horizontal dimension of the gate.

Thrust due to salt water $= 12\frac{1}{2} \times 1.025 \times 62\frac{1}{2} \times 25\, a$ lb. wt.
($62\frac{1}{2}$ is the density of fresh water in lb. per c. ft.)

Thrust due to fresh water $= \frac{1}{2}\, x \times 62\frac{1}{2} \times xa$ lb. wt.

By problem these two thrusts are equal.

Hence $x = 25.3$ ft.

5. A triangular lamina having an area of 4 sq. ft. has its vertices immersed at depths of 1, 2 and 3 ft. respectively in water; find the thrust on the area, the atmospheric pressure being taken as 14·7 lb. per sq. in. [M.U., B.A.]

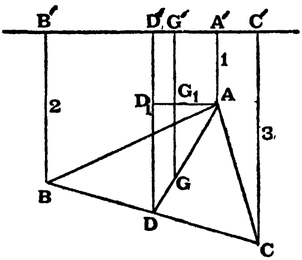

Fig 15.

Let ABC be the lamina with A, B and C at depths of 1, 2 and 3 ft. respectively. Draw AD the median, and let G be the C.G. of the lamina. Draw AA′, BB′, CC′, DD′, GG′, verticals to meet the liquid surface.

Now $DD' = (3 + 2)/2 = 2\frac{1}{2}$ ft.

From A draw AG_1D_1 horizontally. Then $G_1G/D_1D = AG/AD = 2/3$. $\therefore\ G_1G = \frac{2}{3}\, D_1D = \frac{2}{3}\,(2\frac{1}{2} - 1) = 1$ ft.

$\therefore\ GG' = 1 + 1 = 2$ ft.

$\therefore$ Pressure at G = atmospheric pressure + pressure due to liquid $= (14.7 \times 12 \times 12 + 2 \times 62.5)$ lb. wt. per sq. ft.

$\therefore$ Thrust on lamina $= (14.7 \times 12 \times 12 + 2 \times 62.5)\, 4$
$= 8967$ lb. wt.

QUESTIONS ON CHAPTER I

(1) A rectangular lamina is immersed in water with its plane vertical and a side in the surface. Show how to divide the surface into three parts by horizontal lines, so that the thrusts on the parts may be equal.

(2) A square lamina ABCD is immersed in water with the side AB in the surface. Draw a straight line through A so that it divides the lamina into two parts, the thrusts on which are equal.

(3) A triangle is immersed in a liquid with its base in the surface. Divide the area into two parts by a horizontal straight line so that the thrusts on the two parts are equal.

(4) A square lamina of side a is immersed in a vertical position with one side in the surface of a liquid of density ρ_1, which rests on a heavier liquid of density ρ_2. If the depth of the upper liquid is b, find the thrust on the square; and show that, if the thrusts on the two portions of the square in contact with the two liquids are equal, then

$$\rho_1 b (3b - 2a) = \rho_2 (a - b)^2.$$

(5) A vessel in the form of a hollow cone, supported with its axis vertical and vertex downwards, is filled with a liquid. Find the resultant thrust of the liquid on the surface of the cone.

(6) A conical wine glass is filled with a liquid and placed in an inverted position upon a horizontal table. Prove that the resultant thrust of the liquid on the glass is two-thirds of the thrust upon the table.

(7) A hemispherical bowl is filled with a liquid and placed inverted with its circular edge in contact with a horizontal table. Prove that the resultant thrust of the liquid on its surface is one-third of the thrust on the table.

(8) The densities of two liquids are 2ρ and 3ρ respectively. The lighter rests on the heavier to a depth of 4 in. A square of side 6 in. is immersed in a vertical position with one side in the upper surface of the lighter liquid. Show that the thrusts on the portions of the square in the two liquids are as 8 : 11. [M.U., BA., Sep. 1933]

(9) Calculate the total thrust on one side of a rectangular vertical dock gate 45 ft. wide immersed in sea water to a depth of 30 ft., given that 1 c. ft. of sea-water weighs 1025 ozs.

If there is fresh water on the other side of the dock gate, find its depth so that the resultant thrusts on the two sides may be equal. [M.U., B.A., March 1934]

(10) An isosceles triangle, immersed vertically in a liquid, has its base horizontal and vertex in the surface. Divide it by a horizontal line into two parts on which the resultant thrusts are equal. [M.U., B.A., March 1936]

(11) Find the resultant thrust on any plane area immersed in a liquid at rest.

A triangular lamina having an area of 4 sq. ft. has its vertices immersed at depths of 1, 2 and 3 ft. respectively in water; find the resultant thrust on the area, the atmospheric pressure being taken as 14·7 lb. wt. per sq. in.

[M. U., B.A., March 1937]

(12) Find the resultant thrust on a plane area immersed in a fluid.

A rectangle is immersed vertically in a liquid with one side in the surface. How would you divide the rectangle by a horizontal line so that the thrusts on the two parts are equal? [M.U., B.A., Sep. 1938]

(13) Find the resultant thrust on any plane area immersed in a liquid at rest.

A square is placed in a liquid with one side in the surface. Show how to draw a horizontal line in the square dividing it into two portions, so that the thrusts on the two portions are the same. [M.U., B.A., March 1940]

CHAPTER II

CENTRE OF PRESSURE

13. Centre of Pressure.—*The centre of pressure of a plane surface immersed in a fluid is that point in it through which passes the resultant of the thrusts on the various elements of area into which the whole surface may be divided.*

Let AB (Fig. 16) be a plane surface immersed in a a fluid. The surface can be divided up into a number of elements such as dS. A thrust acts on each element normally and hence the thrusts on all the elements are parallel. The point in AB through which the resultant of all these (parallel) thrusts passes is called its *centre of pressure.* The difference between *centre of gravity* and centre of pressure is that the former refers to *equal* forces acting on equal elements, whereas the latter refers to equal elements acted upon by forces *varying* as the depths of the elements. Hence the centre of pressure will always be lower than the centre of gravity.

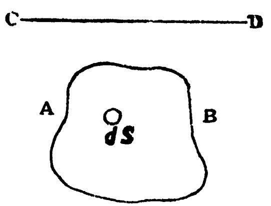

Fig. 16.

14. Centre of Pressure Vertically below Centre of Gravity of Superincumbent Liquid.—It was proved in Art. 12 that the resultant vertical thrust on any surface immersed in a liquid acts through the centre of gravity of the superincumbent liquid. In the case of a plane surface it is obvious that the point through which the resultant thrust passes (*i.e.*, the centre of pressure) is the same as the point through which the resultant *vertical* thrust

passes; for, on every element the resultant thrust and its vertical component are in the same ratio, the angle between the two being constant. Hence *the centre of pressure of any plane area immersed in a liquid is vertically below the centre of gravity of the superincumbent liquid.*

15. General Method of Finding the Centre of Pressure.—Let AB be any plane surface of area S immersed vertically in a liquid of density ρ. Divide up the area into a number of elements such as dS (Fig. 16). Let the depth of this be h below CD, the line in which the plane of the surface meets the liquid level. The thrust on $dS = h\,\rho\,g\,dS$, and the moment of this about CD $= h^2\,\rho\,g\,dS$. The sum of the moments of the thrusts on all such elements $=$ $\int h^2\,\rho\,g\,dS$. This must be equal to the moment of the resultant thrust about CD. The resultant thrust is $\int h\,\rho\,g\,dS$. If H is the depth of the centre of presure below CD, the moment of the resultant thrust $= H\int h\,\rho\,g\,dS$, since, by definition, the centre of pressure is the point through which this resultant thrust passes. Hence

$$\int h^2\,\rho\,g\,dS = H\int h\,\rho\,g\,dS.$$

$$\therefore\quad H = \frac{\int h^2\,dS}{\int h\,dS}.$$

A little consideration will show that the above equation holds good even if the plane surface is not vertical. The liquid surface is supposed not to be subjected to any pressure (such as atmospheric).

16. Centre of Pressure of a Rectangular Lamina with one Side in the Surface.— Let ABCD (Fig 17) be a rectangular lamina immersed vertically in a liquid of density ρ, with its side AB in the surface of the liquid, not subjected to any external pressure. Let $AB = a$ and $AD = b$. Divide up the surface of the rectangle into a number of infinitesimally narrow strips, each parallel to AB

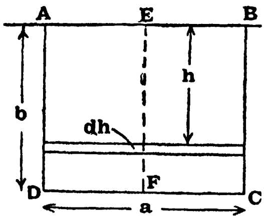

Fig. 17.

and of width dh. Consider the strip at depth h below AB. The area of the strip $= a\,d\,h$, and the thrust on it $= h\rho g \cdot adh$. The moment of this about AB $= h^2\,\rho\,g\,a\,dh$. The sum of the moments of the thrusts on all the strips $= \int_0^b h^2\rho gadh$.

The resultant of the thrusts on all the strips $=$ $\int_0^b h\,\rho\,g\,a\,dh$, and the moment of this resultant about AB $= H \int_0^b h\,\rho\,g\,a\,dh$, where H is the depth of the centre of pressure below AB. Hence

$$\int_0^b h^2\,\rho\,g\,a\,dh = H \int_0^b h\,\rho\,g\,a\,dh$$

$$i.e., \int_0^b h^2\,dh = H \int_0^b h\,dh.$$

$$\therefore\ \tfrac{1}{3}\,b^3 = H.\,\tfrac{1}{2}\,b^2.$$

$$\therefore\ H = \frac{2}{3}\,b.$$

Since the resultant thrust on each strip passes through its mid-point, and since the straight line EF joining the mid-points of AB and CD passes through the mid-points of all the

strips, it follows that the total resultant thrust passes through a point in EF. Hence the centre of pressure of the rectangle lies in the straight line joining the mid-points of the top and bottom sides at depth of $\frac{2}{3}$ b below the top side.

The above result also holds good for any parallelogram with a side in the surface ; b here represents the altitude.

[*Alternative Proof* :— Divide the parallelogram into a number of indefinitely narrow equal strips by drawing lines parallel to AB at equidistant intervals. The thrust on each strip is proportional to the area of the strip and to its depth below the surface, and it may be taken to act at the mid-point of the strip. Since the areas of the strips are the same, the centre of pressure is the centre of a number of parallel forces uniformly increasing in proportion to the depth below AB. This is obviously the centroid of the triangle EDC. Hence the centre of pressure is in EF at a distance of $\frac{2}{3}$ EF from E.]

17. Centre of Pressure of Triangle with a Vertex in the Surface and the Opposite Side Horizontal.— Let ABC be a triangular lamina (Fig. 18) immersed vertically in a liquid of desity ρ with its vertex A in the surface of the liquid, not subjected to any external pressure, and the opposite side BC horizontal.

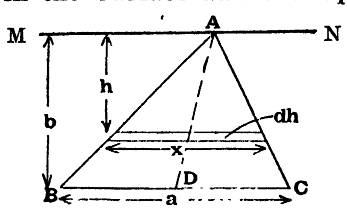

Fig. 18.

Draw MN through A parallel to BC. Let BC $= a$ and the altitude b. Divide up the surface of the triangle into a number of indefinitely narrow strips, each parallel to BC and of width dh. Consider the strip of length x at depth h below A. The thrust on it is $h\,\rho\,g.\ x\,dh$, and the moment of this about MN is $h^2\,\rho\,g\,x\,dh$. But $x/a = h/b$, hence $x =$

$a\,h\,/\,b$. Thus the moment becomes $ah^3\rho g dh\,/\,b$. The sum of the moments due to all the strips $= \int_0^b ah^3\,\rho g\,dh\,/\,b$.

The resultant of the thrusts on all the strips $=$ $\int_0^b h\rho g\,x\,dh = \int_0^b a\,h^2\,\rho\,g\,dh\,/\,b$, and the moment of this about MN $= H\int_0^b ah^2\rho g dh\,/\,b$ where H is the depth of the centre of pressure below MN. Hence

$$\int_0^b ah^3\,\rho g\,dh\,/\,b = H\int_0^b ah^2\,\rho g\,dh\,/\,b$$

$$i.e., \int_0^b h^3\,dh = H\int_0^b h^2\,dh.$$

$$\therefore\ \tfrac{1}{4}\,b^4 = H.\,\tfrac{1}{3}\,b^3$$

$$\therefore\ H = \frac{3}{4}\,b.$$

Since the resultant thrust on each strip passes through its mid-point, and since the median AD passes through the mid-points of all the strips, it follows that the resultant of the thrusts on all the strips passes through a point in AD. Hence the centre of pressure lies in the median AD at a (vertical) depth of $\tfrac{3}{4}b$ below the liquid surface, *i.e.*, at a distance of $\tfrac{3}{4}$ AD from A.

[*Alternative Proof*:— Divide the triangle into a number of indefinitely narrow strips of equal breadth by drawing lines parallel to BC. The thrust on each strip is proportional to its area and to its depth below A and may be taken to act at its mid-point, which must be in AD, the median. Since the area of a strip is proportional to its distance from A measured along AD, as also its depth, the centre of pres-

sure is the centre of a number of parallel forces uniformly increasing in proportion to the *square* of their distance from A. This is obviously the centre of gravity of a pyramid whose vertex is A and the centre of gravity of whose base is D. Hence the centre of pressure is in AD at a distance of $\frac{3}{4}$ AD from A.]

18. Centre of Pressure of a Triangle with one Side in the Surface.— Let ABC (Fig. 19) be a triangular lamina immersed vertically in a liquid of density ρ with the side BC in the surface of the liquid, not subjected to any external pressure. Let BC $= a$ and the altitude $= b$. Divide up the surface into a number of indefinitely narrow strips, each parallel to BC and of width dh. Consider the strip of length x at depth h below BC. The thrust on it is $h\rho g.\ xdh$ and the moment of this about BC is $h^2\rho g\ xdh$. But $x/a = (b-h)/b$, hence $x = a\,(b-h)\,/\,b$. Thus the moment becomes $ah^2\,(b-h)\,\rho g\,dh\,/\,b$. The sum of the moments due to all the strips $=$

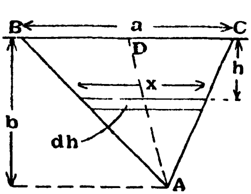

Fig. 19.

$$\int_0^b ah^2\,(b-h)\,\rho g\,dh\,/\,b.$$

The resultant of the thrusts on all the strips $=$

$\int_0^b h\rho g\,x\,dh = \int_0^b ah\,(b-h)\,\rho g\,dh\,/\,b$ and the moment of this about BC $= H\int_0^b ah\,(b-h)\,\rho g\,dh\,/\,b$, where H is the depth of the centre of pressure below BC. Hence

$$\int_0^b ah^2\,(b-h)\,\rho g\,dh\,/\,b = H\int_0^b ah(b-h)\,\rho g\,dh\,/\,b$$

$$i.e., \int_0^b h^2 (b-h)\, dh = H \int_0^b h (b-h)\, dh$$

$$\therefore \left[\tfrac{1}{3}bh^3 - \tfrac{1}{4}h^4 \right]_o^b = H \left[\tfrac{1}{2} h^2 b - \tfrac{1}{3} h^3 \right]_o^b$$

$$\therefore \tfrac{1}{3} b^4 - \tfrac{1}{4} b^4 = H \left(\tfrac{1}{2} b^3 - \tfrac{1}{3} b^3 \right)$$

$$\text{Hence} \quad H = \frac{1}{2} b.$$

Since the resultant thrust on each strip passes through its mid-point, and since the median AD passes through the mid-points of all the strips, it follows that the resultant of the thrusts on all the strips passes through a point in AD. Hence the centre of pressure lies in the median AD at a (vertical) depth of $\frac{1}{2} b$ below BC, *i. e.*, at a distance of $\frac{1}{2}$ AD from D or A.

[*Alternative Proof* :— Divide the triangle into a number of indefinitely narrow strips of equal width by drawing lines parallel to BC. The resultant thrust on each strip is clearly in AD, the median. Consider two strips such that the distance of one from D is equal to that of the other from A (Fig. 20). Let E and F be their mid-points. The thrust on the upper strip is proportional to its area and its depth conjointly, *i. e.*, to AE $\times$ ED. Similarly, the thrust on the lower strip is proportional to AF $\times$ FD. Since AF = ED and, consequently, AE = FD, the two thrusts are equal. Therefore, the resultant of these two thrusts passes through the mid-point of FE, which is also the mid-point of AD. Similarly, the resultants of all other similar pairs pass through this mid-point. Hence the centre of pressure of the triangle is the mid-point of AD.]

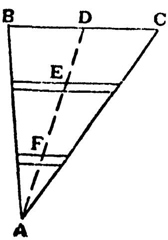

Fig. 20.

19. Centre of Pressure of a Circle.— Let a circular lamina of radius r be fully immersed vertically in a liquid of density ρ. Let O be the centre of the circle (Fig. 21), AB the vertical diameter and h the depth of O below the surface of the liquid, not subjected to any external pressure. Construct a hemisphere with the circle as the base. The liquid contained in this hemisphere is in equilibrium under the action of the following forces:—

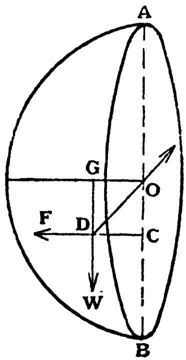

Fig. 21.

(1) Its weight $\frac{2}{3}\pi r^3 \rho g$, which may be taken to act through the centre of gravity G vertically down. We know that $OG = \frac{3}{8} r$ in the radius drawn at right angles to the base.

(2) The thrust on the circle equal to $h \rho g . \pi r^2$, which may be taken to act through C, the centre of pressure.

(Thrust on plane surface = pressure at C. G. $\times$ area.)

(3) The resultant thrust on the curved surface, which must act through O, since the thrusts on the elements of the curved surface, being normal, are all directed towards O.

(These three forces must meet at a point, D).

Taking moments about O, we get

$$\tfrac{2}{3}\pi r^3 \rho g \times OG = \pi r^2 h \rho g \times OC, \text{ where } OG = \tfrac{3}{8} r.$$

Hence $$OC = r^2 / 4h.$$

20. Effect of Further Immersion on Position of Centre of Pressure.— Let F be the resultant thrust on a surface of area S immersed in a liquid of density ρ. Let C_1 be the centre of pressure (Fig. 22). Let the surface be now lowered (without rotation) through a depth z. The effect of

this is to increase the pressure at every point by $z\rho g$, and this is equivalent to an additional thrust of $z\rho gS$ passing through the centre of gravity G of the surface. The new centre of pressure, therefore, is the point C_2 through which passes the resultant of F at C_1 and $z\rho gS$ at G. The centre of pressure is thus shifted nearer the centre of gravity by further immersion. At infinite depth the centre of pressure coincides with the centre of gravity.

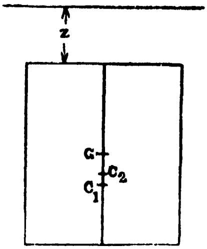

Fig. 22.

In the previous articles the positions of the centres of pressure have been arrived at on the assumption that the surface of the liquid is not subjected to any pressure, such as the atmospheric pressure. If the atmospheric pressure, acting on the liquid, be taken into account, the problem becomes similar to further immersion considered above, P (the atmospheric pressure in absolute units) being substituted for $z\rho g$.

21. Resultant Horizontal Thrust on a Surface.—Let PRQS be the perimeter of any surface (plane or curved) immersed in a liquid (Fig. 23). It is required to find the **resultant horizontal thrust** on it in any given direction. Through each point in the perimeter draw horizontal lines PA, SD, QB, RC, etc., in the given direction, and let these lines meet a verti-

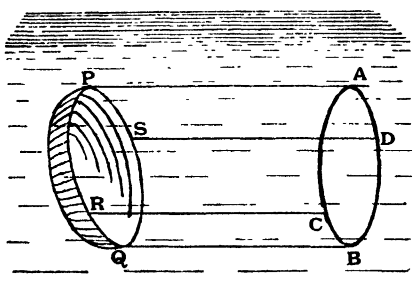

Fig. 23.

cal plane perpendicular to the given direction in a closed curve ADBC.

Imagine the liquid between PRQS and ACBD to be composed of a number of infinitely thin horizontal cylinders. Considering the equilibrium of any such cylinder, and resolving the forces horizontally (cf. Art. 7), we see that the horizontal component of the thrust on the element of the given surface is equal to the horizontal thrust on the element of the surface of projection ACBD. Taking all the cylinders into account, we see that the resultant horizontal thrust on the given surface in the direction PA is equal and opposite to, and in the same line of action as, the resultant (horizontal) thrust on ACBD. The point in ACBD through which the latter resultant passes is, by definition, its centre of pressure. Hence *the resultant horizontal thrust in any given direction on any surface immersed in a liquid at rest is equal to the thrust on the projection of the surface upon a vertical plane perpendicular to that direction, and passes through the centre of pressure of that projection.*

22. Resultant Thrust on Any Surface.—To find the resultant thrust on any surface immersed in a liquid at rest, we must know

(1) The resultant vertical thrust on the surface (Art. 12).

(2) The resultant horizontal thrust in any (horizontal) direction (Art. 21).

(3) The resultant horizontal thrust in the (horizontal) direction at right angles to (2).

These three directions (Fig. 24) are at right angles to one another.

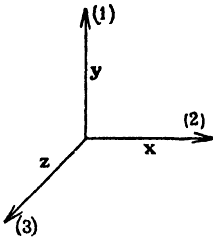

Fig. 24.

The required resultant thrust on the surface is found by the application of the *Parallelopiped of Forces*, used to compound forces in three dimensions, just as the Parallelogram of Forces is used to compound forces in two dimensions, *i.e.*, in a plane.

Here $R^2 = x^2 + y^2 + z^2$.

EXAMPLES

1. A vertical rectangular dock-gate ABDE is hinged so as to rotate about the upper horizontal edge AB. What force must be applied to the middle point of DE to keep the door shut if AB = 10 ft. and AE = 12 ft. and if the water (fresh) rise to the level of AB ?

Thrust on gate = pressure at C. G. × area = $6 \times 62\frac{1}{2} \times 12 \times 10 = 45{,}000$ lb. wt.

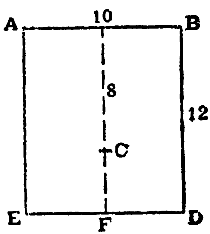

Fig. 25.

This may be taken to act at the centre of pressure C (Fig. 25) of the surface ; C is at a depth of $12 \times \frac{2}{3}$ or 8 ft. below AB.

The moment of the thrust about AB = $45{,}000 \times 8$.

Let x be the force required at F. Then the moment of this about AB = $x \times 12$.

Equating the moments, we get

$$45{,}000 \times 8 = x \times 12, \text{ from which}$$

$$x = 30{,}000 \text{ lb. wt.}$$

2. The water on one side of a lock-gate has a depth of a ft. On the other side the depth is b ft. Prove that the resultant thrust acts at a height of $(a^2 + ab + b^2) / 3(a + b)$ ft. above the bottom of the gate.

[M.U., B.A. & B.Sc., March 1932]

Let the breadth of the gate be c ft. and let ρ represent the density of the water. The thrust on the first side $= \frac{1}{2} a\rho g \times ac$; this may be taken to act at A (centre of pressure) distant $\frac{1}{3} a$ from O (Fig. 26).

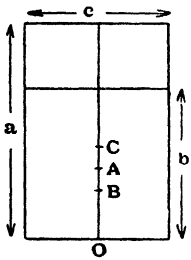

Fig. 26.

The thrust on the second side $= \frac{1}{2} bg\rho \times bc$; this may be taken to act at B distant $\frac{1}{3} b$ from O.

These two forces are in opposite directions. Hence the resultant thrust $= \frac{1}{2} a\rho g \times ac - \frac{1}{2} b\rho g \times bc$. Let this act at C distant y from O.

Taking moments about O,

$$\tfrac{1}{2} a\rho g \times ac \times \tfrac{1}{3} a - \tfrac{1}{2} b\rho g \times bc \times \tfrac{1}{3} b = (\tfrac{1}{2} a\rho g \times ac - \tfrac{1}{2} b\rho g \times bc)\, y.$$

i.e., $\quad a^3 - b^3 = 3y(a^2 - b^2)$.

$\therefore \quad y = (a^2 + ab + b^2) / 3(a + b)$.

3. Find the centre of pressure of a trapezium immersed vertically in a liquid with one of the parallel sides in the surface.

[M.U., B.SC., March 1938]

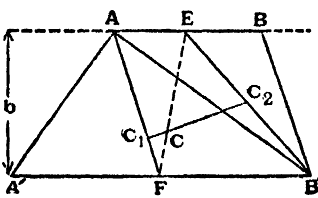

Fig. 27.

Let ABB′A′ be the trapezium with AB in the surface (Fig. 27).

Let AB $= a$, A′B′ $= a'$ and the altitude $= b$. Let ρ be the density of the liquid.

Bisect AB and A′B′ at E and F and join EF.

Clearly the centre of pressure lies in EF. Join AB′, AF and B′E.

Thrust on $\triangle$ AA′B′ $= \frac{2}{3}\, b\rho g \times \frac{1}{2} a'b = \frac{1}{3} a' b^2 \rho g$, which may be taken to act at C_1 at depth $\frac{3}{4}\, b$ below AB (Art. 17).

Thrust on $\triangle$ ABB′ $= \frac{1}{3}\, b\rho g \times \frac{1}{2}\, ab = \frac{1}{6} ab^2 \rho g$, which may be taken to act at depth $\frac{1}{2}\, b$ below AB (Art. 18).

The resultant thrust on the whole trapezium $=$ $\frac{1}{3}\, a'\, b^2\, \rho g + \frac{1}{6}\, ab^2 \rho g$. Let this act at C, the required centre of pressure, at depth x below AB.

Taking moments about the axis AB,

$$\frac{1}{3}\, a'b^2\rho g \times \frac{3}{4}\, b + \frac{1}{6}\, ab^2\rho g \times \frac{1}{2}\, b = (\tfrac{1}{3}\, a'b^2\, \rho g + \tfrac{1}{6}\, ab^2\, \rho g)x.$$

Hence $$x = \frac{b\,(3a' + a)}{2\,(2a' + a)}$$

4. A rectangular lamina whose sides are 12 in. and 18 in. is immersed in water, so that the shorter side is on the surface and the longer side vertical. Find the position of its centre of pressure when the atmospheric pressure is taken into account, the height of the water barometer being 33 ft.

[M.U., B.A. & B.Sc., March 1928]

Thrust due to water $= \frac{3}{4} \times 62\frac{1}{2} \times 1 \times 1\frac{1}{2}$ lb. wt., which may be taken to act at $18 \times \frac{2}{3}$ in. or 1 ft. from the top side.

Thrust due to atmosphere $= 33 \times 62\frac{1}{2} \times 1 \times 1\frac{1}{2}$ lb. wt., which may be taken to act at 9 in. or $\frac{3}{4}$ ft. from the top side.

Let the resultant of these two forces pass through a point at distance y ft. from the top side.

Taking moments about top side,

$$y\,(\tfrac{3}{4} \times 62\tfrac{1}{2} \times 1\tfrac{1}{2} + 33 \times 62\tfrac{1}{2} \times 1\tfrac{1}{2}) = \tfrac{3}{4} \times 62\tfrac{1}{2} \times 1\tfrac{1}{2} \times 1 + 33 \times 62\tfrac{1}{2} \times 1\tfrac{1}{2} \times \tfrac{3}{4}.$$

Hence $y = 34 / 45$ ft. or $9\frac{1}{15}$ in.

$\therefore$ The centre of pressure is on the central vertical line at a distance of $9\frac{1}{15}$ in. from the top side.

5. Shew that if a lamina, always totally immersed in a liquid, be moved without rotation, the vertical distance between the centre of pressure and the centre of gravity varies inversely as the depth of the centre of gravity.

For the sake of simplicity, imagine the lamina to be vertical.

Let h be the depth of the centre of gravity G of the area; let y be the distance, measured vertically downwards, of any indefinitely small area ds below G. Then by definition of G, $\int y\, ds = 0$.

The thrust on $ds = (h+y)\, \rho g\, ds$, where ρ is the density of the liquid.

$\therefore$ Thrust on whole area $= \int (h+y)\, \rho g\, ds$.

Let Y be the vertical distance of the centre of pressure below G.

Equating the moment of the resultant thrust to the algebraic sum of the moments of the thrusts on the various elements, about the horizontal axis through G in the plane of the figure,

$$Y \int (h+y)\, \rho g\, ds = \int y\, (h+y)\, \rho g\, ds,$$

$$i.e.,\ Y \int (h+y)\, ds = \int y\, (h+y)\, ds.$$

$$i.e., \quad Yh\int ds + Y\int yds = h\int y\,ds + \int y^2\,ds$$

$$\text{But} \int y\,ds = 0.$$

$$\therefore \; Y = \int y^2\,ds \Big/ h\int ds$$

$\therefore$ Y varies inversely as h

A little consideration will show that the above proposition is true even if the lamina is not vertical.

N.B. In Arts, 15, 16, 17 and 18 we derived the depth of the centre of pressure in each case on the assumption that the lamina is immersed *vertically*. It can be easily proved, however, that the equations remain the same even if the lamina is inclined at an angle (θ) with the vertical, provided H and h denote the distances *along the plane of the lamina* from the line of intersection of the plane with the surface of the liquid. For, the *depth* of element ds here becomes $h \cos \theta$ and the thrust on the element is consequently $h \cos \theta\, \rho g\, ds$. The moment of this thrust about the line of intersection is $h^2 \cos \theta\, \rho g\, ds$. By the principle of moments,

$$\int h^2 \cos \theta\, \rho g\, ds = H\int h \cos \theta\, \rho g\, ds.$$

Thus $\cos \theta$ cancels out and the equation remains the same as before in every case.

QUESTIONS ON CHAPTER II

(1) A vertical rectangular dock-gate ABCD is hinged so as to rotate about the upper horizontal edge AB, and is fastened at the mid-point of the lower edge CD. If AB = 10 ft. and BC = 15 ft., calculate in tons weight the stress on the fastener in CD, given that the dock is full of water.

(2) A triangle is vertically immersed in a liquid with its base in the surface. Show that a horizontal straight line drawn through the centre of pressure of the triangle divides it into two parts, so that the thrusts on them are equal.

(3) A plane rectangular lamina ABCD is immersed in a liquid of density ρ and kept vertical with the side AB in the free surface. Find the positions of the centres of pressure of the two triangles into which the rectangle is divided by the diagonal AC. [M.U., B.A. & B.Sc., Sep. 1927]

(4) Prove that the horizontal line drawn through the centre of pressure of a rectangle, one side of which is in the surface, divides the rectangle into two parts, the thrusts on which are as 4 : 5. [M.U., B.A , March 1933]

(5) Define 'centre of pressure'. Find the centre of pressure of a triangle immersed in a liquid with one side in the surface.

The water on one side [illegible] rectangular flood-gate, 12 ft. high and 9 ft. wide, r[illegible]he top, and on the other side to half the height. Find [illegible]sition and magnitude of the resultant thrust on the gate. [M.U , B.A., Sep. 1937]

(6) Define 'centre of pressure'.

Find the centre of pressure of a triangle immersed in a liquid with a vertex in the surface, and the opposite side horizontal.

A dock-gate is 12 ft. wide. There is fresh water on one side of the gate to a depth of 9 ft. and on the other side to a depth of 6 ft. Find the resultant force on the gate and the point of action. [M.U., B.A.. March 1938]

(7) Obtain an expression for the centre of pressure of a plane area immersed in a fluid.

Find the centre of pressure of a trapezium immersed vertically in a liquid with one of the parallel sides in the surface. [M.U., B.Sc., March 1938]

(8) Find the centre of pressure of a rectangular lamina when immersed in a liquid with one side in the surface.

A lock-gate 12 ft. wide has fresh water weighing 62·5 lb. per cubic foot on one side to a depth of 10 ft. and sea water weighing 64 lb. per cubic foot to a depth of 6 ft. on the other. Calculate the resultant thrust on the gate and find the position where it acts.

[M. U., B. A., March 1939]

(9) Find the centre of pressure of a triangular lamina immersed in a liquid with a vertex in the surface and the opposite side horizontal.

The depth of water on one side of a rectangular lock-gate is 12 ft and on the other side it is 4 ft. The breadth of the gate is 10 ft. Find the magnitude and line of action of the resultant thrust on the gate. 1 cubic foot of water weighs 62·5 lb. [M. U., B. A., Sep. 1939]

(10) (*a*) How is the resultant thrust on a curved surface determined?

(*b*) Define 'centre of pressure.' Find the centre of pressure of a triangle immersed in a liquid with a vertex in the surface and the opposite side horizontal.

[M. U., B. Sc., Sep. 1939]

(11) Define 'centre of pressure.'

Find the centre of pressure of a triangular lamina immersed with its vertex in the surface and the base horizontal within the liquid.

How does the position of the centre of pressure vary as the vertex is gradually lowered within the body of the liquid? [M. U., B. Sc., March 1940]

CHAPTER III

FLOATING BODIES

23. Resultant Thrust on a Body wholly or partly immersed in a Liquid.—Let a body be wholly or partly immersed in a liquid at rest (Fig. 28). Imagine the body to be removed, and the cavity (supposed undisturbed meanwhile) filled with extra liquid. The liquid added will, of course, be in equilibrium. The forces acting on the liquid added (called the *replacing liquid* or the *displaced liquid*) are

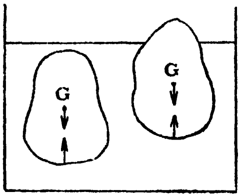

Fig. 28.

(1) its weight, which may be taken as acting through its C. G. vertically down and (2) the resultant hydrostatic thrust. Hence these two forces must be equal and opposite and in the same line. Now, the hydrostatic thrust must depend only on the depth, shape and area of the surface, not on the nature of the body. Hence the resultant hydrostatic thrust on the original body also must be equal to the weight of the liquid filling the cavity, *i. e.*, to the weight of the displaced liquid and must pass vertically up through the centre of gravity of the displaced liquid.

24. Condition of Floatation — We have seen that, if a body is wholly immersed in a liquid, the resultant hydrostatic thrust acts vertically upwards and is equal to the weight of the liquid displaced by the body, *i. e.*, to the weight of an equal volume of the liquid. This principle is known as **Archimedes' principle**, and the resultant hydrostatic thrust is called the **force of buoyancy**. When the weight of the body fully immersed is less than the weight of the liquid displaced, *i. e.*, is less than the weight of an equal

volume of the liquid, the resultant hydrostatic thrust will be greater than the weight of the body, and hence the body will float. Therefore, the condition for floatation is that *the weight of the body must be less than the weight of an equal volume of the liquid, i.e., the density of the body as a whole must be less than the density of the liquid.*

25. Conditions for Equilibrium of a Floating Body.—When a body is floating in a liquid, the forces acting on the body are (1) the weight of the body, which may be taken to act through its C. G. vertically downwards and (2) the resultant hydrostatic thrust. For equilibrium, these two forces must be equal, opposite and in the same straight line. We have already seen (Art. 23) that the resultant hydrostatic thrust is equal to the weight of the liquid displaced and passes vertically upwards through the C. G. of the displaced liquid. Hence for equilibrium of a floating body, the following two conditions must be satisfied :—

(1) *The weight of the displaced liquid must be equal to the weight of the floating body.*

(2) *The centre of gravity of the body and that of the displaced liquid must be in the same vertical line.*

[The proposition of Art. 12 may also be applied here instead of that of Art. 23.]

The same method of proof may be extended to the case of a body floating with one part of its volume immersed in one liquid and another part in another liquid. In this case the sum of the weights of the displaced liquids must be equal to the weight of the body and the vertical line through the centre of gravity of the body must pass through the centre of gravity of the combination of the displaced liquids in their own positions. This includes the case of a body

floating in a liquid exposed to air, when the buoyancy of the air is taken into account.

[In the above discussion we have used the term 'displaced liquid' in the sense of 'the liquid which would fill the space occupied by the immersed portion of the body'. But the term is rather inappropriate. For example, it is possible for a body to be immersed in a liquid, the total volume of which may be less than the volume of the body. Here there is no question of the body having 'displaced' its own volume of the liquid. The term *replacing liquid* would, therefore, be more appropriate. But we have used the term 'displaced liquid' throughout this book on account of its universal acceptance, and the reader must know exactly what it means.]

26. Weight of a Body in Air.—A body whose real weight (*i.e.*, weight in vacuum) is W and whose density is ρ will experience in air a force of buoyancy equal to the weight of its own volume of air, *i.e.*, $(W / \rho) \times d$, where d is the density of air. Hence the *apparent weight* of the body in air $= W - Wd/\rho = W(1 - d/\rho)$. If this body is weighed in a common balance using 'weights' whose density is ρ' and if the total real weight (in vacuum) of the 'weights' used for balance is W', then equating the apparent weights of the two, we have

$$W(1 - d/\rho) = W'(1 - d/\rho').$$

$$\text{Hence } W = W'(1 - d/\rho')/(1 - d/\rho).$$

Since d/ρ and d/ρ' are small in comparison with 1, their squares and higher powers and their product can be ignored in comparison with 1. Hence

$$W = W'(1 - d/\rho')(1 + d/\rho)$$
$$= W'(1 - d/\rho' + d/\rho).$$

27. Stability of Equilibrium of a Floating Body.— We know that if a body, freely suspended, is in equilibrium, the centre of gravity of the body must be vertically below or above the point of suspension. But the equilibrium itself will be *stable* or *unstable* according as the centre of gravity (G) of the body is respectively below (Fig. 29a) or above (Fig. 29 b) the point of suspension (S). Similarly, a floating body will be in *equilibrium* if the two conditions given in Art. 25 are satisfied, but the equilibrium may or may not be *stable*. For stability of equilibrium another condition must be satisfied. We shall first consider the definitions of a few terms used in this connection.

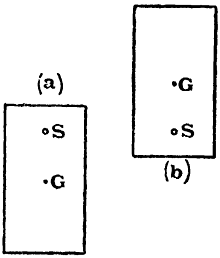

Fig. 29.

The centre of gravity of the displaced liquid is called the **centre of buoyancy**.

The section in which the surface of a liquid cuts a body floating in it is called the **plane of floatation**. In

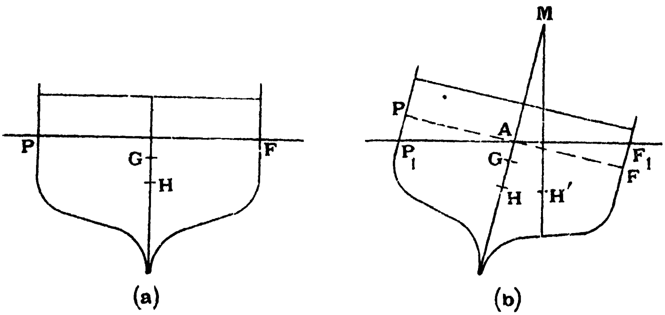

Fig. 30.

Fig. 30, PF and P_1F_1 represent the planes of floatation in (a) and (b) respectively.

If a floating body move about so that it takes up in succession every position in which the volume of the liquid displaced by it remains constant, the locus of the centre of buoyancy is called the **surface of buoyancy**.

If a body floating freely be slightly turned, so that the same quantity of liquid as before is displaced, the point in which the vertical line through the new centre of buoyancy meets the line joining the centre of gravity of the body to the original centre of buoyancy is called the **metacentre**.

Explanation.—If a body floats in equilibrium in a liquid (Fig. 30 a) the centre of gravity G of the body and the centre of buoyancy H are in the same vertical line (Art. 25). Let the line HG be considered fixed in the body. Now turn the body through a small angle, in such a way that the mass of the liquid displaced remains the same. Then if the vertical through the new centre of buoyancy H′ (Fig. 30 b) intersect HG, the point of intersection M is called the *metacentre.*

The distance GM between the centre of gravity and the metacentre is called the **metacentric height**.

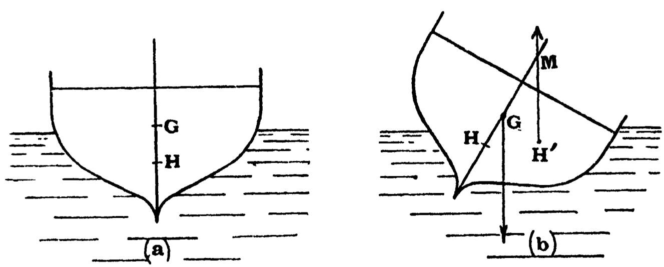

Fig. 31.

We shall now discuss the condition for stability. If the metacentre M is *above* the centre of gravity G as in Fig. 31,

the weight of the body, which may be taken to act through G vertically down, and the force of buoyancy acting vertically up through H′ tend to bring the disturbed body back to the original position. Hence the equilibrium here is stable. But if the metacentre is *below* the centre of gravity (Fig. 32), the two forces tend to turn the disturbed body further from the original position. Hence the equilibrium here is unstable. Therefore, *for stability of equilibrium, the metacentre must be above the centre of gravity.*

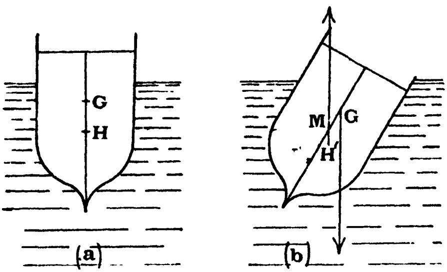

Fig. 32.

28. **Distance (HM) between the centre of Buoyancy and the Metacentre.**— If a floating body turns slightly from its equilibrium position, the original plane of floatation PF and the subsequent plane of floatation P_1 F_1 (Fig. 30) meet in a line (represented by A in the figure), which we may call the **axis of rotation**. It can be shown that

$$\text{HM} = \frac{Ak^2}{V},$$

where Ak^2 is the moment of inertia of the plane of floatation about the axis of rotation and V the volume of liquid displaced.

It must be noted that it is GM that is called the metacentric height and not HM.

29. **Experimental Determination of the Metacentric Height of a Ship.**— The metacentric height of a ship may

be determined by the following method, which is known in naval architecture as 'the inclining experiment'.

The mass of the ship M' is found from its displacement and a mass m of several tons, which is already on board the ship, is moved across the deck through a measured distance l ($=$ A B, Fig. 33). Consequently, the ship turns through an angle θ (a very small one, of course). This angle of inclination is found by means of a plumb-bob suspended in the ship.

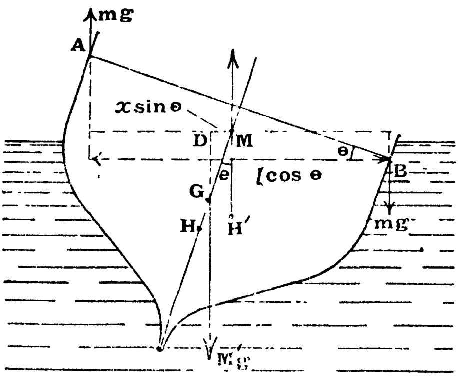

Fig. 33.

Let H and H′ be the original and altered centres of buoyancy, G the centre of gravity of the ship and M the metacentre. Let x be the metacentric height GM. Then taking moments about M, we have

$$M'g\, x \sin\theta = mg\, l \cos\theta.$$

$$\therefore\ x = ml \,/\, M' \tan\theta = ml \,/\, M'\, \theta,$$

since θ will usually be small.

[(1) Transferring a weight mg from A to B is equivalent to: retaining mg at A and introducing two new forces, each equal to mg, one vertically up at A and the other vertically down at B. This latter system is again equivalent to a weight of $M'g$ at G, a force mg acting up at A and another force mg acting down at B.

(2) The moment of the force of buoyancy about M is zero, since the force passes through M.

(3) The metacentric height GM will remain the same only when the ship is loaded in the same manner as when tested.

(4) Instead of moving a heavy weight across the deck of the ship, two boats on the deck at distance l apart may be alternately filled with the same mass of water.|

30. Metacentre of Floating Body with immersed Part Spherical.— If the portion of the floating body which is immersed in the liquid is spherical, the centre of curvature C of this spherical portion must be the metacentre. For, the thrust at each element of the spherical surface is normal to the surface and so passes through C: hence the resultant thrust passes through C. We have already seen (Art. 23) that the resultant thrust passes vertically through the centre of buoyancy, and this applies not only to the equilibrium position, but also to the disturbed position. In the equilibrium position, the resultant thrust passes also through the centre of gravity of the body. Hence HG in the equilibrium position and the vertical through H′ in the disturbed position both pass through C, which is therefore the metacentre.

31. The Common Hydrometer.—This is a variable immersion hydrometer graduated and ready for rapid use. It is used for the determination of the specific gravities of liquids. It is usually made of glass and consists of a uniform, thin, graduated stem AD (Fig. 34 a), ending in a comparatively large bulb and another smaller bulb, weighted with mercury, below it. The weighted bulb is for making the hydrometer float vertically.

As this is a constant weight hydrometer, the lighter the liquid in which it floats, the greater the immersion. The instrument may be graduated as follows :—

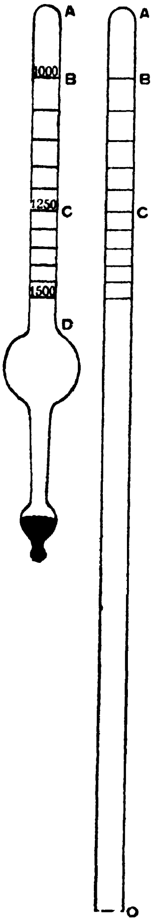

Fig. 34.

Float the hydrometer in water and measure the length of the stem l_1 above the water surface. Then float it in another liquid of known density ρ, and measure the length of the stem l_2 above the liquid surface.

Let V be the volume of the whole hydrometer and a the area of cross-section of the uniform stem. Then the weight of the hydrometer = the weight of the liquid displaced in each case =

$(V-l_1 a) \times 1 = (V-l_2 a) \times \rho$, from which

$V=a\,(l_2\,\rho-l_1)\,/\,(\rho-1)$.

Now $(l_2\,\rho-l_1)\,/\,(\rho-1)$ is a quantity which can be calculated from the data obtained. Represent it by k.

Then $V=k\,a$.

Let the hydrometer be now floated in a liquid of unknown density x. Let l be the length of the stem above this liquid. Then, as before,

$(ka-l_1a) \times 1 = (ka - la)\,x$, from which

$x = (k - l_1)\,/\,(k - l)$ or $k - l = (k - l_1)\,/\,x$.

Thus for successive values of x the corresponding values of l on the stem can be calculated or vice versa.

It is found that when the density increases uniformly the graduations get closer and closer. This relation can be investigated more clearly as follows :—

Let BA be the portion of the stem above water and CA the portion above a liquid of density ρ, when the hydrometer is floated in the two liquids successively. Imagine the uniform stem AD to be continued to O (Fig. 34 b) so that the volume of OA is equal to V, the volume of the whole hydrometer. Then

$V = OA.a$ (a being the area of cross-section), and the weight of the hydrometer = the weight of the liquid displaced in each case =

$$(OA.\,a - BA.\,a) \times 1 = (OA.\,a - CA.\,a)\,\rho$$

$$\therefore\ \rho = \frac{OB}{OC} \qquad \text{or} \qquad OC = \frac{OB}{\rho}.$$

Hence the graduations corresponding to densities (or specific gravities) in arithmetical progression are at distances from a point in the stem produced which are in harmonic progression and vice versa.

EXAMPLES

1. A man whose weight is 150 lb. and whose volume, exclusive of the head, is 2 c.ft. can just float in water with his head above the surface by the aid of a lump of cork completely immersed. If the sp. gr. of the cork is 0·24, find the volume of the cork. (1 c.ft. of water weighs $62\frac{1}{2}$ lb.)

Let x be the volume of the cork in c. ft.

Its weight $= x \times 0{\cdot}24 \times 62\frac{1}{2}$ lb.

Weight of water displaced by man and cork =

$(2 \times 62\frac{1}{2} + x \times 62\frac{1}{2})$ lb.

Equating total weight of man and cork to weight of water displaced by both,

$150 + x \times 0{\cdot}24 \times 62\frac{1}{2} = 2 \times 62\frac{1}{2} + x \times 62\frac{1}{2}$,

from which $x = 10/19$ c.ft.

2. A solid homogeneous cylinder of height h and density ρ floats with its axis vertical in a liquid of density ρ_1. A liquid of density ρ_2 is gently poured into the vessel till the top of the cylinder is covered. Calculate what part of the cylinder is immersed in the liquid of density ρ_2, if it continues to float with its axis vertical. ($\rho_1 > \rho_2$.)

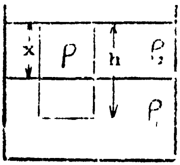

Fig. 35.

Let A be the area of cross-section of the cylinder. Then the volume of the cylinder is $A\,h$ and its weight $A\,h\,\rho$.

Let x be the immersion in the upper liquid. Then the immersion in the lower liquid is $h - x$ (Fig. 35).

Equating the weight of the body to the sum of the weights of the liquids displaced, we get

$$A\,h\,\rho = A\,x\,\rho_2 + A\,(h - x)\,\rho_1.$$

Hence $x = h\,(\rho_1 - \rho)\,/\,(\rho_1 - \rho_2)$

3. A small piece of wood floats half immersed in fresh water exposed to the atmosphere. How much of it will be immersed in the fresh water inside a diving bell, 10 ft. high, lowered till its top is 47 ft. below the surface of the water? The height of the water barometer is 34 ft. and the density of air at atmos. pressure and temp. is d. Take the temperature to be the same throughout.

First let us calculate the density of the air in the bell. Let x be the height of the air column in the bell. Then by Boyle's law,

$$10 \times 34 = x\,(34 + 47 + x)$$

from which $x = 4$ ft.

$\therefore$ Pressure of air inside bell $= 34 + 47 + 4 = 85$ ft. of water.

$\therefore$ Density of air inside bell $= d \times 85/34 = d \times 5/2$.

Let V be the volume of the wood and W its weight.

Let y be the fraction of its volume immersed in the water under the bell. Then

$$W = \tfrac{1}{2} V \times 1 + \tfrac{1}{2} V \times d \text{ (in open atmos.), and}$$
$$W = Vy \times 1 + V(1-y)\, 5\, d/2 \text{ (in the bell)}$$
$$\therefore \quad Vy + V(1-y)\, 5\, d/2 = V/2 + Vd/2.$$

Hence $y = (1 - 4\,d) / 2\,(1 - 5\,d/2) = (2 - 3\,d)/4$ approximately.

4. A cylinder of sp. gr. 0·95 and length 16 in. floats with its axis vertical in a vessel of water. Find the height to which oil of sp. gr. 0·84 should be poured in order that the cylinder may be just fully immersed.

[M. U., B. A., March 1939]

Let A be the area of cross section of the cylinder and x the height of oil required in inches.

Then $16A \times 0{\cdot}95k = xA \times 0{\cdot}84\,k + (16-x)\,A \times 1 \times k$, where k is a constant.

Hence $x = 5$ in

5. A cylinder of radius a, height h and weight W floats with its axis vertical in a liquid, of twice its density, contained in a cylinder of radius b. Show that the work necessary to depress the cylinder until it is just immersed is $\tfrac{1}{4} W h\,(1 - a^2/b^2)$. [M.U., B.Sc., March 1932]

The cylinder floats half immersed, since its density is half of that of the liquid.

Now if the cylinder is pressed down so as to be submerged 1 cm. further, the displacement of the liquid $= \pi a^2 \times 1$, and the rise of liquid level $= \pi a^2 / \pi b^2 = a^2 / b^2$.

So the top of the cylinder has descended only $1 - a^2/b^2$.

Hence for immersion of $h/2$, the top of the cylinder must be pushed down through $\tfrac{1}{2} h\,(1 - a^2/b^2)$.

Force to be applied in the beginning $= 0$; force at the end $= 2\,W - W = W$.

$\therefore$ Average force applied $= W/2$.
Work done $=$ average force $\times$ distance $=$
$\frac{1}{2} W \times \frac{1}{2} h\ (1 - a^2/b^2) = \frac{1}{4}\ Wh\,(1 - a^2/b^2)$.

6. Find the condition for stability in the case of a uniform circular cylinder, of radius r, height h, and density a, floating with its axis vertical in a liquid of density b.

Let the area of cross section of the cylinder πr^2 be represented by A. Let h' be the depth of immersion (Fig. 36).

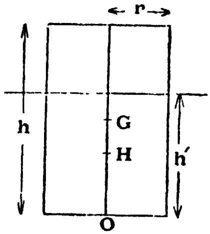

Fig. 36.

Equating the weight of the floating body to the weight of the liquid displaced, we get

$$Aha = Ah'b.$$
$$\therefore\ h' = ah/b.$$

For stability HM $>$ HG.

Now $\text{HM} = Ak^2 / V = A\frac{r^2}{4} \Big/ Ah' = \frac{r^2}{4} \Big/ \frac{a}{b}h = r^2 b / 4\,ah$, and

$\text{HG} = \text{OG} - \text{OH} = \frac{1}{2} h - \frac{1}{2} h' = \frac{1}{2} h - \frac{1}{2}\ ah / b =$
$\frac{1}{2}\ h\ (b - a) / b$.

$\therefore\ \ r^2 b / 4\, a\, h > \frac{1}{2}\, h\ (b - a) / b$

$$\text{i.e., } r^2 > 2\ h^2 \left(\frac{a}{b} - \frac{a^2}{b^2}\right).$$

7. Find the condition for stability of a cone floating in water with its axis vertical and vertex downwards.

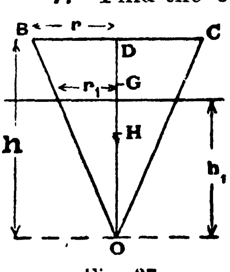

Fig. 37

Let OBC be a section of the cone through the axis OD (Fig. 37). Let $\text{OD} = h$, $\text{BD} = r$. Let h_1 be the depth of immersion, r_1 the radius of the plane of floatation and s the density of the cone.

Now $HM = Ak^2 / V = Ak^2 / \frac{1}{3} Ah_1 = 3r_1^2 / 4h_1$.
But $r_1 / h_1 = r / h$, and

$$\tfrac{1}{3} \pi r_1^2 h_1 = \tfrac{1}{3} \pi r^2 hs \text{ (Art. 25), i.e., } r_1^2 h_1 = r^2 hs$$

$\therefore \quad h_1^3 = h^3 s$, i.e., $h_1 = h \sqrt[3]{s}$.

Hence $HM = 3r_1^2 / 4h_1 = 3r^2 \sqrt[3]{s} / 4h$,

and $HG = OG - OH = \frac{3}{4} h - \frac{3}{4} h_1 = \frac{3}{4} h (1 - \sqrt[3]{s})$.

For stability of equilibrium, $HM > HG$.

i.e., $3r^2 \sqrt[3]{s} / 4h > 3h (1 - \sqrt[3]{s}) / 4$.

i.e., $r^2 / h^2 > (1 - \sqrt[3]{s}) / \sqrt[3]{s}$.

8. In H. M. S. Achilles, a ship of 9000 tons displacement, it was found that moving 20 tons from one side of the deck to the other, a distance of 42 ft., caused the bob of a pendulum 20 ft. long to move through 10 in. Prove that the metacentric height was 2·24 ft. [Math. Tripos, 1884]

Metacentric height $= ml / M'\theta$ (Art. 29)

$$= \frac{20 \times 42}{9000 \times 10/240} = 2{\cdot}24 \text{ ft.}$$

9. A thin cylinder, one end of which is rounded off in the form of a hemisphere, floats with its spherical end partly immersed. If the body is hollow and of uniform thickness throughout, find the greatest height of the cylinder consistent with stable equilibrium. [M. U., B.Sc., Sep. 1933]

A 'thin cylinder' means a cylinder with thin sides and bottom. Let r be the radius.

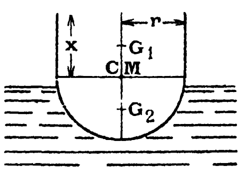

Fig. 38

Here the centre of curvature C (Fig. 38) is the metacentre M (Art. 30).

Let x be the maximum height of the cylinder above C for stability. In this limit of stability, the C. G. of the whole body coincides with M.

The C. G. of the cylindrical part (G_1) lies at a distance of $\frac{1}{2}$ x above C, and its mass is $\propto 2\ \pi\ r\ x$.

The C. G. of the hemispherical part (G_2) lies at a distance of $\frac{1}{2}$ r below C, and its mass is $\propto 2\pi\ r^2$.

Hence if C is to be the C G. of the combination,

$$2\pi\ r\ x \times \tfrac{1}{2}\ x = 2\ \pi r^2 \times \tfrac{1}{2}\ r.$$

$$\therefore\ x = r.$$

10. A common hydrometer floats in a liquid of density ρ_1 with a length l_1 of its stem exposed. A length l_2 is exposed when it floats in a liquid of density ρ_2. Determine the density of the liquid in which the hydrometer will float with a length l of the stem exposed.

Let V be the volume of the whole hydrometer and a the area of cross-section of the stem. Then the weight of the hydrometer =

$$(V-al_1)\ \rho_1 = (V-al_2)\ \rho_2 = (V-al)\ \rho,$$

where ρ is the density required

From the first two expressions we get

$$V\ (\rho_1-\rho_2) = a\,(l_1\ \rho_1 - l_2\ \rho_2) \quad \text{or}$$

$$V = a\ (l_1\rho_1 - l_2\rho_2)/\ (\rho_1 - \rho_2).$$

Substituting this in the first and third,

$$\left\{a\ (l_1\rho_1-l_2\rho_2)/\ (\rho_1-\rho_2) - al_1\right\}\rho_1 =$$

$$\left\{a\ (l_1\rho_1- l_2\rho_2)/\ (\rho_1-\rho_2) - al\right\}\ \rho, \text{ from which}$$

$$\rho = \rho_1\rho_2\,(l_1-l_2)/\ [(l_1 - l)\ \rho_1 + (l-l_2)\ \rho_2]$$

QUESTIONS ON CHAPTER III

(1) A solid (not hollow) cylinder, one end of which is rounded off in the form of a hemisphere, floats with its spherical end partly immersed. Find the greatest height of the cylinder consistent with stability of equilibrium.

(2) Account for the fact that the graduations on the stem of a common hydrometer are not equidistant.

A cylindrical piece of cork of height h is floating with its axis vertical in a basin of water. If the basin be placed under the receiver of an air-pump and the air be pumped out, prove that the cork will sink through a distance, $d\,(1-s)\,h\,/\,(1-d)$, where d and s are respectively the specific gravities of air and cork.

[M.U., B.A. & B.Sc., March 1929.]

(3) Show that a uniform circular cylinder of specific gravity $\frac{1}{2}$ cannot be in stable equilibrium when floating upright in water if its length is greater than $\frac{3}{4}$ of its diameter.

[M. U., B.A. & B. Sc., April 1930].

(4) Calculate the volume of hydrogen which a balloon must contain in order that its total lifting power may be equal to 750 lb. wt., the weight of 100 c. ft. of air being 8 lb. and the sp. gr. of hydrogen referred to air 0·07.

[M.U., B.A., March 1933]

(5) A cube of sp. gr. s floats in water with four edges vertical. Find when the equilibrium is stable.

[M.U., B.A., Sep. 1933]

(6) A solid cone floats with its axis vertical and vertex downwards in a liquid of four times its own specific gravity. Prove that for stability of equilibrium the semi-vertical angle of the cone should exceed $37°28'$.

[M.U., B. Sc., March 1934]

(7) A cylinder of wood (sp. gr. s) of height h and radius r is floating in water with its axis vertical. Show that, for stability of equilibrium, r / h should be greater than $\sqrt{2s(1-s)}$. [M.U., B. Sc., Sep. 1935]

(8) The whole volume of a common hydrometer is 15 c. c., and its stem is 3 mm. in diameter. The hydrometer floats in a liquid A with 3 cm. of its stem above the surface, and in another liquid B with 6 cm. above the surface. Compare the densities of the liquids.

[M.U., B.A., Sep. 1936]

(9) A cylinder of radius r and height h floats with its axis vertical in a liquid of twice its own density. Find the condition for stability of equilibrium.

[M.U., B. Sc., Sep. 1936]

(10) Derive the conditions for the equilibrium of a floating body ; discuss the condition for stability.

A cylinder of wood, 6 ft. long, floats with its axis vertical in water and 4 ft. immersed. Find the least possible radius of its base for it to remain in stable equilibrium.

[M.U., B. Sc., March 1937]

(11) Discuss the conditions of equilibrium of a body floating freely in a liquid. What is the condition for stability ? [M.U., B.A., Sep. 1937]

(12) Derive the conditions of equilibrium of a body floating freely in a liquid. What is the condition for stability?

How is the metacentric height of a ship experimentally determined ? [M.U., B. Sc., Sep. 1938]

(13) Explain the conditions of equilibrium of a body floating freely in a liquid.

Give the theory of the common hydrometer.

A cylinder of sp. gr. 0·95 and length 16 in. floats with its axis vertical in a vessel of water. Find the height to which oil of sp. gr. 0·84 should be poured in order that the cylinder may be just fully immersed.

[M.U., B.A., March 1939]

(14) Find the conditions of equilibrium and stability of a body floating in a liquid.

Describe how the metacentric height of a ship may be determined experimentally. [M.U., B.A., March 1940]

(15) Describe in detail the method of graduating the stem of a common hydrometer.

The lengths of the stem of a common hydrometer exposed to the air, when it is dipped in liquids of specific gravities 1·12 and 1·25, are 2 in. and 5 in. respectively. Find the sp. gr. of a liquid in which the hydrometer floats with 3 in. of its stem exposed. [M.U., B. Sc., March 1940]

CHAPTER IV

THE ATMOSPHERE AND BAROMETER

32. The Atmosphere. — Our atmosphere of air surrounds the earth to a considerable height (about 100 miles) from the surface. Since air has weight, the atmosphere exerts a considerable pressure upon the surfaces exposed to it. This pressure, called the *atmospheric pressure*, is about 15 lb. wt. per sq. in. or 1 kg. wt. per sq. cm. near the sea level. The pressure of the atmosphere is less on a mountain than at sea-level, since the height of the air column in the former case is less. Similarly, the atmospheric pressure is greater in a mine than at sea-level. As in the case of a liquid the air exerts pressure in all directions, upwards, sideways, and downwards, and the pressure at a point is the same in all directions. There is, however, one important distinction between the two. The pressure due to a liquid at a point is proportional to the depth of the point from the surface level and the difference in pressure between two points is proportional to the difference in level between the points. In the case of the atmosphere, however, the difference in pressure between two points is not simply proportional to the difference in level between the points, for, while the density of a liquid is practically independent of pressure, that of a gas varies considerably with pressure owing to its high compressibility. Small differences in level can, however, be calculated from the difference in pressure between the two points and the mean density of the air within the range. ($P_1 - P_2 = h \rho g$).

33. The Barometer. — A barometer is an instrument for measuring the atmospheric pressure. The first barometer

was constructed by Torricelli by filling a long tube (about 80 cm. long), closed at one end and open at the other, with mercury and inverting it vertically into a trough of mercury. The mercury in the tube was found to occupy only a height of about 76 cm. above the mercury level in the trough. By the principle of balancing columns, this height must be a measure of the atmospheric pressure.

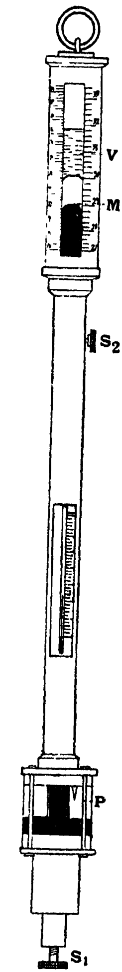

Fig. 39
Fortin's barometer.

34. Fortin's Barometer.—In an ordinary barometer described above two readings are required, corresponding to the lower and higher levels of mercury, before the atmospheric pressure can be determined. In Fortin's barometer (Fig. 39), the lower level of mercury can be brought to a fixed position corresponding to the zero of the scale by an adjustable bottom (Fig. 40) provided for the cistern, and thus only one reading need be taken. The air inside the cistern communicates with the outside air through the lid of the cistern. The cistern is closed at the bottom with a piece of chamois leather which can be raised or lowered by the screw S_1. The zero of the scale of the barometer corresponds to the point of the ivory pin P and, before reading the instrument, the screw S_1 is turned

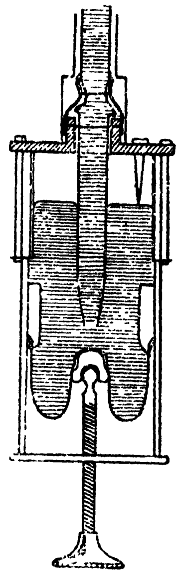
Fig. 40
Cistern with adjustable bottom.

until the mercury level in the cistern just touches the point of the pin P, as indicated by the point of the pin just touching that of its image. Then the screw S_2 is turned until the lower edge (*i.e.*, zero) of the vernier V is just in a level with the mercury surface M in the tube, seen through two glass windows opposite to one another. A plate at the back connected to the vernier moves with the vernier and the lower edges of this plate and the vernier are in the same horizontal plane when the barometer is fixed vertically. When adjusting the vernier, the eye must be in the line joining these two lower edges, so as to avoid the error of parallax. Then the main scale reading (just below the zero of the vernier) and the vernier reading are taken and the height of the mercury column is thus obtained. The barometer reading may vary not only from day to day but also during the same day. A moderate fall in the barometer (about one inch) will indicate coming rain or wind. A considerable fall (1·5 inches or more) indicates coming storm.

35: Corrections for the Reading of Fortin's Barometer.—The reading of Fortin's barometer is corrected and standardised as follows :—

(1) *Correction for expansion of scale.* If h is the reading at t°C. on the scale constructed to read correctly at 0°C., then the actual height of the mercury column is $h(1+at)$, where a is the coefficient of linear expansion of the material of the scale.

(2) *Correction for expansion for mercury.* Let $h(1+at)$ be the actual height of the mercury column at t°C. Let h° be the height which the mercury column would occupy at 0°C. Let ρ_0 and ρ_t be the densities of mercury at 0°C. and t°C. respectively and c its coefficient of cubical expansion. Then the atmospheric pressure $= h(1+at)\,\rho_t\, g = h_0\,\rho_0\, g$.

$$\text{Hence } h_0 = h(1+at)\,\rho_t\,/\,\rho_0 = h(1+at)\,/\,(1+ct)$$
$$= h(1+at)\,(1-ct) = h\,[1-(c-a)\,t] \text{ nearly.}$$
$$a = 0{\cdot}000019 \text{ for brass and } c = 0{\cdot}000182.$$

(3) *Correction for variation of g.* The barometric height at a place where the acceleration due to gravity is g must be standardised to the value at sea-level in latitude 45,° where the acceleration due to gravity (g_0) is 980·6 cm./sec^2. For this, the barometric height must be multiplied by g/g_0. It can be shown that

$$g/g_0 = 1 - 0{\cdot}00259 \cos 2\lambda - 1{\cdot}96\, l \times 10^{-9},$$

where λ is the latitude of the place and l its height in centimetres above sea-level.

(4) *Correction for capillarity effect.* Owing to capillarity effect, the mercury is depressed, and so the correction is always positive. The narrower the bore, the greater is this depression. The correction is found by comparison with a barometer of very wide bore (more than 2·5 cm. in diameter), in which this effect is negligible, and is usually supplied by the maker of the instrument. For a tube of diameter ¼ in. or 6 mm. the correction will be about + 1 mm.

(5) *Correction for vapour pressure of mercury.* At 20°C. the vapour pressure of mercury is only about 0·001 mm. of mercury; so this correction (+0·001 mm.) may be ignored at ordinary temperatures.

36. The Barometric Liquid.—Mercury is generally used as the barometric liquid for the following reasons :—

(1) It is the densest liquid available ; hence the height of the mercury barometer will be the shortest possible. A very tall barometer causes difficulties in construction and reading, and is not easily portable. A water barometer, for

example, will be about 1034 cm. (i.e., $76 \times 13{\cdot}6$ cm.) or 34 ft. (i.e., $30 \times 13{\cdot}6$ in.) in height.

(2) The difficulty of correcting for the vapour pressure, which varies with temperature, is avoided in the mercury barometer, since the vapour pressure of mercury at ordinary temperatures is negligibly small. With water, the vapour pressure is considerable (17·5 mm. of mercury at 20°C.).

(3) Mercury does not wet glass.

(4) It is clearly visible in a glass tube since it is opaque.

(5) It is one of the liquids which can be obtained pure.

The disadvantage of mercury is that, on account of its high density, the height of its column in the barometer will not change much for small changes in the atmospheric pressure; that is, the mercury barometer is not sensitive.

If slight variations of atmospheric pressure are to be indicated, a *glycerine barometer* must be used. The height of the glycerine barometer will be about 818 cm.

37. The Aneroid Barometer.—This instrument, as its name implies, contains no liquid. It consists of a small chamber closed by a diaphragm of thin corrugated metal and partially exhausted (Fig. 41). This diaphragm, attached to a spring, is thrust in or out a little according as the external pressure increases or decreases, the extent of motion being proportional to the

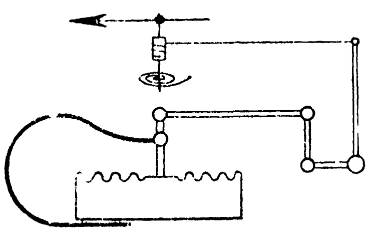

Fig. 41.
Principle of the aneroid barometer.

change of pressure. This motion is magnified by a system of levers and transmitted to an index. The instrument is graduated by direct comparison with a standard mercury barometer. Its indications, however, are not quite reliable. Its chief advantages are portability and sensitiveness.

38. The Normal Atmospheric Pressure.—The *normal atmospheric pressure* is defined as being equivalent to the pressure exerted by a vertical column of mercury 76 cm. in height at 0°C. where g is 980·6 cm. per sec. per sec. It is equal to $76 \times 13{\cdot}596$, *i.e.*, 1033 gm. wt. per sq. cm; or $76 \times 13{\cdot}596 \times 980{\cdot}6$, *i.e*, $1{\cdot}013 \times 10^6$ dynes per sq. cm.; or 14·7 lb. wt. per sq. in.

The density of the air becomes less and less as we ascend higher and higher above the sea-level. The height which the normal atmosphere would occupy if it had a uniform density of 0·001293 gm. per c.c. is called the **height of the homogeneous atmosphere**. If H represents its value, then $H \times 0{\cdot}001293 = 76 \times 13{\cdot}596$, from which

$$H = 7990 \text{ metres.}$$

In the F. P. S. system, $H \times 0{\cdot}001293 \times 62{\cdot}4 = \frac{30}{12} \times 13{\cdot}596 \times 62{\cdot}4$, from which $H = 26{,}280$ ft. (nearly 5 miles).

39. The Gas Laws.—(1) Temperature being constant, the volume of a given mass of gas is inversely proportional to its pressure, *i.e.*, T being constant $V \propto 1 / P$ (Boyle's Law).

(2) Pressure being constant, the volume of a given mass of gas is directly proportional to the absolute temperature, *i.e.*, P being constant, $V \propto T$ (Charles Law).

(3) Volume being constant, the pressure of a given mass of gas is directly proportional to the absolute temperature, *i.e.*, V being constant, $P \propto T$.

40. The Gas Equation.—We can embody the three gas laws in one equation called the *gas equation.*

By Boyle's law, $V \propto \frac{1}{P}$ when T is constant.

By Charles' law, $V \propto T$ when P is constant.

$\therefore \quad V \propto T / P$ when both T and P vary.

That is, PV / T is constant for a given mass of a gas. This constant is represented by R when the mass of the gas is 1 gm. Thus we get

$$PV = R\,T.$$

Corollaries.—(1) Since $P_1 \ V_1 \ / \ T_1 = P_2 \ V_2 / T_2$ and $V \propto \frac{1}{\rho}$ for the same mass, $P_1 \ / \ \rho_1 \ T_1 = P_2 \ / \ \rho_2 \ T_2$.

(2) If we have two portions of gas, P_1, V_1 and T_1 referring to the first portion and P_2, V_2 and T_2 referring to the second portion, then on mixing the two portions

$$P_1 \ V_1 \ / \ T_1 + P_2 \ V_2 \ / \ T_2, = PV \ / \ T,$$

where P, V and T refer to the mixture. This can be proved as follows :—Let V_1' and V_2' be the volumes which the first and second portions of gas would respectively occupy at pressure P and absolute temperature T. On mixing the portions now the total volume would be $V_1' + V_2'$, since they are at the same pressure and temperature. But the total volume at P and T is V. Therefore $V_1' + V_2' = V$. Now

$$P_1 \ V_1 / T_1 = PV_1' / T \text{ and } P_2 \ V_2 / T_2 = PV_2' / T$$

$$\therefore \quad P_1 \ V_1 / T_1 + P_2 \ V_2 / T_2 = PV_1' / T + PV_2' / T = PV \ / \ T.$$

41. Vitiated Vacuum in a Barometer.— A barometer which contains some air in the tube will, of course, give faulty readings. To test if there is air in a barometer tube, screw up the bottom of the cistern (Fortin's) or incline the

tube sufficiently. At a certain stage the whole tube will be filled with mercury if there is no air in it. But if there is any air in it, a bubble of air will always be left, however much we may incline the tube or screw up the bottom.

It is possible to determine the correct atmospheric pressure with such a faulty barometer by the application of Boyle's law. Let h_1 be the height of the mercury column (above the mercury level in the cistern, in all cases) and l_1 the length of the air column in the tube. Now raise or depress the tube or lower or raise the bottom of the cistern sufficiently so that the air column is about double or half of what it was. Note the height of the mercury column h_2 and the length of the air column l_2. Let P be the atmospheric pressure. Applying Boyle's law to the air enclosed in the tube,

$$(P - h_1)\, l_1 = (P - h_2)\, l_2.$$

From this P is determined. To determine the atmospheric pressure with the barometer at any other time, let $(P - h_1)\, l_1 = k$, and let h and l be the height of the mercury column and the length of the air column respectively at the time. Then the pressure of the air enclosed at the time $= k/l$, and the atmospheric pressure $= h + k/l$. The tube is supposed to be uniform and the temperature constant.

42. The Diving Bell. – This is an apparatus for enabling a man to dive to a considerable depth under water in order to examine a sunken vessel, to lay or repair the foundation of a pier, or to pick up pearl-oysters.

The principle of the diving-bell can be understood by inverting a beaker full of air over water and immersing it under the liquid. To whatever depth the beaker may be depressed, the air contained will always remain in it at the

upper part, though its volume will diminish as the depth is increased.

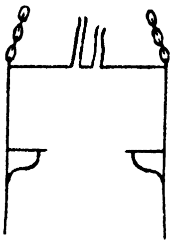

Fig. 42
The diving bell.

The diving-bell (Fig. 42) consists of a large cylindrical vessel, closed at the top and open below, and heavy enough to sink in water with the contained air. There is provision for a person to sit or stand inside the bell, and the air which is always present above enables him to breathe. In modern bells, however, there are two tubes attached to the upper part of the bell, one for pumping in fresh air and the other for withdrawing foul air.

The tension in the chains supporting the bell is equal to the weight of the bell *minus* the weight of water it displaces.

As the bell sinks, the water rises inside, and hence the weight of water displaced becomes less, and the tension in the chains increases.

43. Relation between Height and Pressure. The relation between the height of one place above another and the atmospheric pressures at the two places can be derived as follows: Take two points P and Q (Fig. 43) very near each other in the atmosphere, Q being vertically above P. Let the heights of P and Q above the ground be x and $x + dx$; let the pressures at these points be p and $p + dp$ respectively; and let the average density of the air within PQ be ρ.

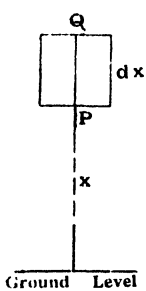

Fig. 43.

Imagine a small cylinder described about the vertical line PQ as the axis. Let its area of cross-section be a.

Resolving the forces vertically, we get for the equilibrium of the cylinder

$$(p + dp)\, a + a\, dx\, \rho\, g = pa,$$

i.e., $dp + \rho\, g\, dx = 0.$

If the air is at constant temperature,

$$\rho / p = \text{a constant} = k, \text{ say.}$$

Then $dp + kpg\, dx = 0,$

i.e., $dp\, /\, p + kg\, dx = 0.$

Integrating, $\log_e p + kgx = \text{a constant}.$

If p_1 and p_2 are the pressures at heights h_1 and h_2,

$$\log p_1 + kgh_1 = \log p_2 + kgh_2,$$

i.e., $\log_e (p_1 / p_2) = kg\, (h_2 - h_1).$

Now $k = \rho / p = 0{\cdot}001293 / 76 \times 13{\cdot}6 \times 980{\cdot}6$

(supposing the air is at $0°$C.),

and $\log_e (p_1 / p_2) = 2{\cdot}3026 \log_{10} (p_1 / p_2).$

Thus the difference in height $h_2 - h_1$ between two places can be calculated from the pressures at the two places.

As this equation has been obtained on the assumption that the temperature and the acceleration due to gravity (g) are both constant throughout, neither of which suppositions is even approximately correct for very large differences in altitude, we cannot apply the equation to very large differences in height.

[If the mean temperature of the air in the range is different from O C., the density of the air at that temperature and normal pressure is to be substituted for 0·001293. It must be remembered that p_1 is the pressure at the *lower* place and p_2 that at the *higher* place.]

Corollary. The equation $\log_e (p_1 / p_2) = kgh$, where $h = h_2 - h_1$, may be written as

$$p_1 / p_2 = e^{kgh}.$$

If p_0 is the pressure at ground level and p the pressure at height h above it and ρ_0 and ρ the corresponding densities then

$$p_0 / p = e^{kgh} \text{ and } \rho_0 / \rho = e^{kgh},$$

since density is directly proportional to pressure according to our assumption that temperature is constant. Hence

$$p = p_0 / e^{kgh} \text{ and } \rho = \rho_0 / e^{kgh}$$

Hence *as the altitude increases in arithmetical progression, the density, as well as the pressure, diminishes in geometric progression.*

EXAMPLES

1. A barometer (of uniform tube) reads 73 cm. when the length of the remaining space above in the tube is 5 cm. The mercury level in the cistern is now raised in relation to the tube, and the barometer reading and the length of the remaining space are now respectively 71 cm. and 3 cm. Find the atmospheric pressure. If the reading in the barometer at any other time (at the same temp.) is h and the length of the remaining space is l, what is the atmospheric pressure then?

The barometer contains air since its reading varies with the length of the remaining space.

Let P be the atmospheric pressure. Then, applying Boyle's law to the air enclosed, $(P-73)\,5=(P-71)\,3$, from which $P=76$ cm.

Also, $k=(76-73)5=15$. $\therefore$ pressure of enclosed air $=15/l$, and the atmospheric pressure at the time $=h+15/l$.

2. If the density of air at N. T. P. is 0·00129 gm. per c.c., calculate the ascent corresponding to a fall of the barometer from 30 to 27 in., the mean temperature being 25°C. [M.U., B. Sc.]

Density of air at normal pressure and 25°C
$= 0{\cdot}00129 \times 273 / 298 = 0{\cdot}00118$ gm per c.c.

$\therefore \quad \rho / p = 0{\cdot}0018 / 76 \times 13{\cdot}6 \times 980 6 = k.$

Now $2\ 3026 \log_{10} (p_1 / p_2) = k\,g\,h$

But $\log_{10} (p_1 / p_2) = \log_{10} (30 / 27) = \log_{10} 30 - \log_{10} 27.$
$= 0{\cdot}0457.$

$\therefore \quad 2{\cdot}3026 \times 0{\cdot}0457 = h \times 0{\cdot}00118 / 76 \times 13{\cdot}6$

Hence $h = 0{\cdot}92 \times 10^5$ cm. or $0{\cdot}92$ kilometre.

3. If the height of the homogeneous atmosphere is 26,000 ft., prove that the altitude above the sea-level of a place at which the barometric height is 26 in., when the reading at the sea-level is 30 in., is about 3718 ft., the change in temperature being neglected. [M. U.. B. Sc.]

Let ρ be the density of air in lb. per c. ft. at N. T. P. Then by definition of height of homogeneous atmosphere (Art. 38), normal atmospheric pressure v in poundals per sq. ft. $= 26{,}000 \times \rho \times 32$.

$\therefore \quad \rho / p = 1 / 26{,}000 \times 32 = k.$

Now $2{\cdot}3026 \log_{10} (p_1 / p_2) = k\,g\,h.$

But $\log_{10} (p_1 / p_2) = \log_{10} (30 / 26) = 0{\cdot}0621.$

$\therefore \quad 2{\cdot}3026 \times 0{\cdot}0621 = h / 26{,}000.$

Hence $h = 3718$ ft.

QUESTIONS ON CHAPTER IV

(1) A uniform glass tube 25 in. long and closed at the upper end is attached to the *sounding lead* of a ship. On drawing the lead from the bottom of the ocean, the tube is found to be wet with water up to 20 in. inside. What is the pressure at the bottom due to the water? Calculate the depth of the ocean at the place, the density of sea-water being 64 lb. per c. ft. (Atm. press., normal.)

(2) Determine the depth to which a cylindrical diving bell 8 ft. high must be lowered in the sea in order that the air in it may be compressed to $\frac{3}{4}$ of its original volume, taking the specific gravities of mercury and sea-water as 13·6 and 1·025 respectively, and the height of the barometer as 30 in. [M.U., B.A. & B. Sc., March 1932]

(3) If the readings of a barometer with an imperfect vacuum are h_1 and h_2 when the true readings are $h_1 + a$ and $h_2 + b$ respectively, find the length of the tube occupied by air in the two cases. Show also that the top of the tube is at a height $(ah_1 - bh_2) / (a - b)$ above the mercury in the reservoir. [M.U., B.A. & B.Sc., Sep. 1932]

(4) A barometer stands at 30 in., and the space occupied by the Torricellian vacuum is 2 in. If a bubble of air which would, at atmospheric pressure, occupy $\frac{1}{2}$ in. of the tube be introduced into the tube, prove that the surface of mercury will be lowered by 3 in. Show also that the height of a correct barometer, when this false one stands at x inches, is $x + 15 / (32 - x)$ in.

[M.U., B.A. & B. Sc., March 1928]

(5) Neglecting the effect of temperature difference, calculate the height of Bangalore above sea-level if the barometric height at Bangalore is 27 in. and at sea-level 30 in. [M.U., B.A. & B.Sc., March 1931]

(6) If the height of the homogeneous atmosphere is 26,000 ft., prove that the altitude above the sea level of a place at which the barometric height is 26 in., when the reading at the sea-level is 30 in., is about 3718 ft., the change in temperature being neglected. [M.U., B.Sc., Sep. 1932]

(7) The readings of a barometer with an imperfect vacuum are 28 and 29 in. when the true readings are 28.5

and 29.75 in. respectively. Prove that the correction to be applied for any other reading x is $3 / (62 - 2x)$.

[M.U., B A., Sep. 1934]

(8) Find an expression for the variation of atmospheric pressure with height, stating the assumptions you make, and show how the result may be used to find the difference of height between two stations.

Find, neglecting variation in temperature, the altitude of a place where the mercury barometer reads 705 mm., when the reading at sea-level is 750 mm.

[MU., B.Sc., March 1937]

(9) Show that the atmospheric pressure diminishes in geometric progression as the height increases in arithmetic progression, assuming the temperature to be constant.

If the density of air at N.T.P. be 0·00129 gm. per c.c., calculate the ascent corresponding to a fall of the barometer from 30 to 27 in., the mean temperature being 25°C.

[M.U., B.Sc., Sep. 1938]

(10) When the heights above the earth's surface in the atmosphere are in arithmetical progression, show that the corresponding pressures are in geometric progression, assuming the temperature to be constant.

Calculate the difference in heights between two stations, if the atmospheric pressures at the places are 63·5 and 75 cm., taking the necessary data from the tables.

[M.U., B.Sc., March 1939]

CHAPTER V

PUMPS AND PRESSURE-GAUGES

Pumps

44. The Compression Air Pump.—This consists of a cylindrical *barrel* fitted with a *piston* P (Fig. 44). There is a valve (V_1) in the piston and another (V_2) in the exit tube, both valves opening only inwards (towards the *receiver* R). In the common bicycle pump, which is an example of this pump, the valve in the piston is a cup valve, and the valve in the pneumatic tube is a tissue valve of rubber.

Now let us push the piston from the outer end inwards. Owing to the increasing pressure of the enclosed air, V_1 is closed tightly and the air at high pressure opens V_2 and passes through into the vessel. On pulling the piston outwards, a vacuum is produced behind it. The air in the vessel, by its pressure, simply closes the valve V_2 tightly. But the air from outside opens the valve V_1 and enters the barrel. Thus, as the cycle of operations is repeated, one barrelful of air at atmospheric pressure is sent into the vessel for each double stroke.

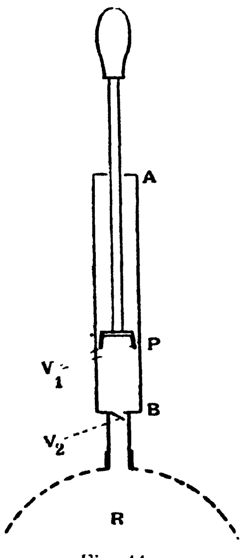

Fig. 44.
The Compression air pump.

Let V be the (constant) volume of the *receiver* (vessel) and v that of the barrel. Then after n strokes a volume nv of air at atmospheric pressure (P) is sent into the receiver. If P_n is the pressure of the air in the receiver at the end, by Boyle's aw, $P\,(nv + V) = P_n V$, since the vessel

originally had a volume V of air at atmospheric pressure. Hence

$$P_n = P\,(nv + V)\,/\,V.$$

The density of air also increases in the same proportion. If d and d_n are the original and final densities respectively, then, by the law of conservation of mass, $d_n V = d\,(nv + V)$.

$$\therefore\ d_n = d\,(nv + V)\,/\,V.$$

If the receiver is a pneumatic tube having no air in the beginning, then $Pnv = P_n V$, where V is the final volume of the receiver.

Applications of Compressed Air. — *Riveting hammers* and *pneumatic tools* for stone cutting, iron chipping, drilling, etc., are driven by compressed air. Other common applications are in *sand blasts* for cleaning metal and stone surfaces and *air-brakes* on electric and steam cars. Compressed air is also supplied to *divers.*

45. The Exhaust Air Pump — Piston Type.—*Hawksbee's Air Pump.* This is the simplest form of the exhaust pump. It consists of a cylindrical *barrel* C (Fig. 45) in which a tight-fitting *piston* P works. The barrel is connected by a tube to the *bed-plate* B on which the bell jar R to be exhausted is inverted with the joint made air-tight with wax. The vessel to be exhausted may also be connected to the tube by means of pressure rubber tubing. In the piston there is a valve (V_1) opening only outwards (away from the vessel) and at the junction between the

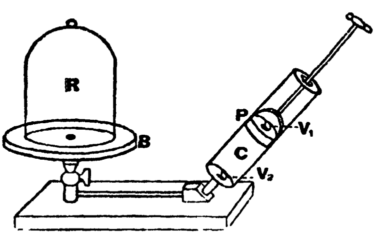

Fig. 45.
Hawksbee's air pump.

barrel and the tube there is another valve (V_2) also opening outwards.

Now let us draw the piston outwards from the lower end of the barrel. A vacuum is produced behind it. The pressure of the air outside closes the valve V_1 firmly, since the valve cannot open inwards. But the air from the vessel R and the connecting tubes opens the valve V_2 outwards and enters the space. Let us now push the piston in from the outer end of the barrel. The air in the barrel is compressed, but it cannot escape in through the valve V_2, which cannot open inwards. Hence it opens the valve V_1 outwards and escapes out. Thus the cycle of operations is repeated. After each double stroke one barrelful of air is removed. In what follows, a 'stroke' is taken to comprise the double movement of the piston, outward and inward.

Let V be the volume of the vessel ('receiver') and tubes and v that of the barrel. Let P be the pressure of the atmosphere, and P_n the pressure of air inside after n strokes. After each pull a volume V of air occupies the volume $V+v$. Hence, by Boyle's law,

$PV = P_1\ (V + v)$ in the first stroke.

$\therefore$ Pressure P_1 after 1 stroke $= PV\ /\ (V + v)$.

By similar successive steps, $P_n = P\ [V\ /\ (V + v)]^n$

Again let d be the density of the air at the beginning and d_n the density after n strokes. Then since a volume V of air occupies the volume $V + v$ after each pull and since the mass of this is constant,

$Vd = (V + v)d_1$ in the first stroke.

$\therefore$ density d_1 after 1 stroke $= dV\ /\ (V + v)$.

By similar successive steps, $d_n = d\ [V\ /\ (V + v)]^n$.

We could have derived this from the previous equation from the law that the density of a gas is proportional to its pressure.

Tate's Air Pump.—In this a double piston P_1, P_2 connected together works in the barrel AB (Fig. 46). A

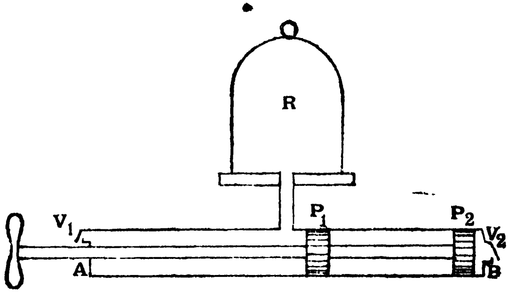

Fig. 46. Tate's air pump.

tube in the middle of the barrel connects the barrel with the 'receiver.' There are two valves V_1 and V_2 opening only outwards, at the ends of the barrel. The principle of the instrument may be easily understood from the figure. As there is no valve between the receiver and the barrel, this pump is more efficient than the previous one.

The exhaustion produced by the piston type of exhaust pumps described above is of the order of 1 mm. of mercury.

[In expressing a pressure by the height of a liquid column, if no liquid is mentioned, mercury is implied.]

46. Defects of Piston Type Exhaust Pumps.—The above-mentioned types of exhaust pumps cannot produce a high degree of exhaustion, owing to the following reasons:—

(1) There is always some leakage in the valves.

(2) The valves cannot open unless there is a certain minimum pressure of air inside.

(3) There will be some minimum space (called the *clearance* or *dead space*), between the piston and the valve-end of the barrel, and this lowers the efficiency of the pump.

For example, let the length of the barrel be 30 cm. and the clearance space $\frac{1}{2}$ mm. Then the valve in the piston (Fig. 45) will open only if the 30 cm. of air, when compressed to $\frac{1}{2}$ mm., is greater than the atmospheric pressure. Therefore, if p is the pressure of the air in the barrel and P the atmospheric pressure, the former becomes $p \times 300/\frac{1}{2}$, *i.e.*, $600\,p$ when compressed by the piston. Thus $600\,p$ must not be less than P, *i.e.*, the minimum value of p is $P/600$ or 1·3 mm. of mercury nearly.

(4) With each stroke we remove only a *certain fraction* of the air which occupied the vessel just before the commencement of *that* stroke.

From the equation, $P_n = P\,[V/(V+v)]^n$ it is clear that we cannot reduce the pressure to a very low value unless the number of strokes is very large. But pumping is not effective beyond the first few strokes on account of the first three defects. Hence for small values of n, for which alone pumping is effective, the pressure cannot reach a very low value.

47. The Filter Pump or the Water-Jet Pump.—This pump (Fig. 47) is usually made of glass. A is connected by a rubber tube to a water pipe and the side tube D to the vessel to be exhausted. The water issues from the nozzle C at a very high speed, and escapes by the tube E carrying away some of the air with it. Consequently, air is sucked from the receiver through the tube D.

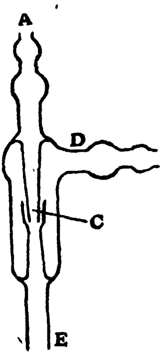

Fig. 47.
Water-jet pump or filter pump.

The exhaustion produced by this pump is not high, being only of the order of 10 mm., but it is convenient and economical, working without any attention at all

48. Uses of High Vacuum.—High vacuum is necessary in the following apparatus:—

(1) *Vacuum discharge tubes*, investigations with which have shed light upon the whole domain of electrical science and even upon the constitution of matter itself.

(2) The *X-ray tube*, used in medicine and surgery, and X-ray spectroscopy.

(3) The *thermionic valve*, used in wireless transmission and reception.

(4) The *mercury vapour lamp* and the *incandescent tungsten lamp* (of low power).

(5) The *photo-electric cell*, used for a variety of purposes, such as in television, Talkies, etc.

(6) The *vacuum spectrograph* used for the investigation of ultra-violet light.

(7) The thermos flask.

49. High Vacuum Technique.—The lowest pressure that the above-mentioned piston-type exhaust pumps can produce is only of the order of 1 mm. To produce still lower pressures, the mercury pumps of **Sprengel** and **Toepler** were formerly used. The former pump was used by Graham, Bunsen, Crookes, etc., and the latter by Sir J. J. Thomson and Lord Rayleigh, in their experimental work. But the operation of these mercury pumps is extremely tedious and protracted. It is recorded that Lord Rayleigh spent a whole morning in exhausting a two-litre globe with a Toepler pump.

While the mercury pumps mentioned above produce the Torricellian vacuum (the vapour pressure of mercury, about 10^{-3} mm.) after laborious work for a long time, the **hyvac pump** conveniently worked by an electric motor can produce the same degree of vacuum in a few minutes.

The highest possible vacua (about 10^{-8} mm.) are now easily and quickly obtained by the **mercury vapour pump** or the **diffusion pump** first devised by **Gaede**. This pump can exhaust a space only if it is maintained at a fairly low pressure (such as 10^{-2} mm.) by another pump, which, in this connection, is called the **backing pump** or the **fore-pump** The hyvac pump is a suitable backing pump for the diffusion pump. This combination can easily produce a vacuum of 10^{-6} mm. in a six-litre vessel in about half an hour.

Mercury or oil used in the diffusion pump, backing-pump oil and grease introduce their own vapour into the exhausted space. Their removal is effected by a **liquid air trap** (Fig. 48). This device consists in cooling some convenient part of the apparatus by liquid air, which freezes out the vapours. When liquid air is not available, a **potassium trap** is used. In this trap, some connected tube in the system is lined inside with pure distilled potassium, which effectively absorbs mercury vapour (but not the oil vapours). Instead of potassium, sodium may also be used. In any case, the trap is subsequently sealed out.

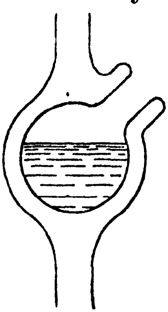

Fig. 48
Liquid air trap.

Apart from using the diffusion pump, a high vacuum can be quickly obtained from a low vacuum by the use of (1) *absorbents* or (2) *'getters'*

The most efficient **absorbent** is **cocoanut shell charcoal** powder. This is introduced in a side tube attached to the receiver system, then heated during the pumping operation, and then cooled with liquid air after connection with the pump is cut off. The charcoal absorbs most of the residual gas in the receiver system. It is estimated that a suitable type of charcoal can absorb about 1000 times its own volume of gas. The charcoal tube is then sealed out.

The residual gas in a moderate vacuum can also be absorbed by means of a "**getter**". This is usually **phosphorus** or **magnesium**, which, on being heated, chemically combines with the gas or gases left. Further a getter introduced into the vessel as fine deposit on the walls forms a steady corrective on further gas emission.

One serious difficulty has been experienced in maintaining a high vacuum in a vessel on account of the slow release of *absorbed* and *adsorbed* gases from glass and metal into the vacuum. In the case of glass, the gases thus liberated are water vapour and carbon di-oxide, and in the case of metals, hydrogen and carbon monoxide. The remedy for this in the case of glass is to heat, while pumping, both the vessel and the connecting tubes to the highest temperature they can stand (400 to 1000°C according to the kind of material) without the walls collapsing. In the case of metals (such as filaments, electrodes, etc.) they are previously heated in vacuo to nearly their melting point for several hours before being introduced into the vessel. This process of removing gas from glass and metal is called '**out gassing**'.

50. Sprengel's Pump.– Sprengel's pump consists of a bent tube as in Fig. 49, with a reservoir A at the top. There is a stop cock S below the reservoir. The vessel to be exhausted is connected to the side-tube D. The lower end E dips into a beaker of mercury.

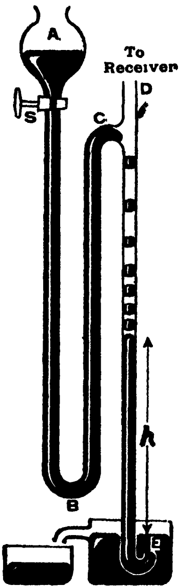

Fig. 49.
Sprengel's pump.

Mercury is poured into the bulb, and the stop cock opened until the portion up to C is filled with mercury. Then the stop cock is adjusted so that the mercury from C passes over in drops to E. Each drop, acting as an air-tight piston, carries down, as it falls, some of the air before it and drives it out at the open end E.

When the exhaustion is fairly complete, each mercury drop, as it falls down the tube strikes the top of the mercury column h with a sharp metallic click. When exhaustion is complete, h must be the barometric height.

The mercury which flows down must be periodically returned to the reservoir.

The fold BC, which is a subsequent improvement over the original type, serves to prevent air from rushing into the receiver when the mercury in the bulb is exhausted. CB, (and E C) must, of course, be more than the barometric height.

This pump ultimately reduces the pressure to the vapour pressure of mercury, *i.e.*, about 10^{-3} mm.

51. The Toepler Pump.—The Toepler pump (Fig. 50) consists of a cylindrical glass vessel A to which are attached two vertical tubes BC and DE each about 80 cm. long. The fall *tube* DE must be about 1 mm. in diameter. BC is connected by a piece of rubber-tubing to a reservoir F containing mercury, which can be raised or lowered. The tube G branching from A is connected to the vessel (receiver) to be exhausted. The end E of the fall tube dips into a vessel of mercury.

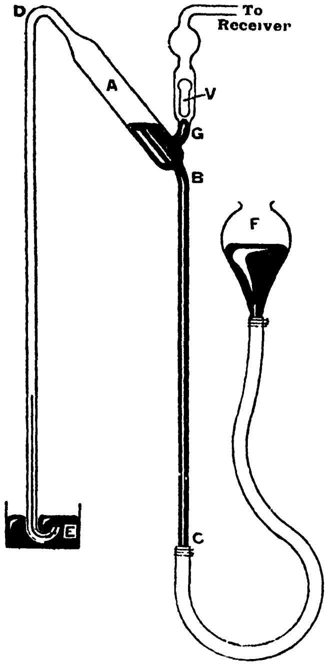

Fig. 50.
Toepler Pump.

The process of working the pump consists in raising and lowering the reservoir F alternately. On raising the reservoir sufficiently, the mercury rises in BA, thereby sealing up the side tube G and driving the gas in A before it down DE and out into the atmosphere. On lowering F, the level of mercury sinks below B, and immediately gas from the receiver rushes and fills the vacuum created in A. During each cycle the pressure of the gas is reduced by the same fraction of its value. The mercury here plays the part of the piston and valves in the ordinary pump (without, however, their defects). It collects in the vessel at E and must be periodically poured back into F.

To prevent the flow of mercury into the receiver, the latter must be at the barometric height above the level D.

This flow may also be prevented by introducing a glass valve V, which allows the apparatus to be made more compact. It must be noted that the function of this valve is different from that in the simple mechanical pumps, and its use does not in any way affect the degree of exhaustion. The function of sealing G is efficiently done by the mercury.

The disadvantage of this pump is that it is laborious and slow in action. But it is simple in principle and there are no leakages, no valves operated by minimum pressure of gas and no clearance space. In course of time, it can reduce the pressure to the vapour pressure of mercury, *i.e.*, about 10^{-3} mm. In these respects it resembles the Sprengel pump.

52. The Cenco "Hyvac" Pump.—The 'hyvac' pump

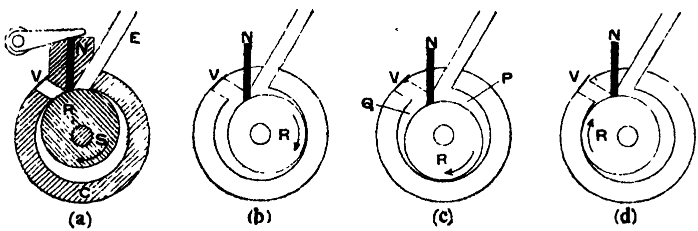

Fig. 51.
The 'hyvac' pump.

is much more rapid and convenient than the mercury pumps described above. It consists of a rotor R (Fig. 51) mounted eccentrically on a shaft S which passes through the centre of a cylindrical casing C. A scraping vane N, which can move in a vertical direction, is kept pressed against the rotor by a spring. The vessel to be exhausted is connected to the tube E. V is an exhaust valve.

As the rotor is rotated in the direction of the arrow, the space P in communication with E increases in volume, while the space Q containing the gas previously drawn from the receiver diminishes in volume. The effect of this is to draw more gas in P from the receiver and to compress and drive out through the valve V the gas previously drawn from the receiver.

Usually two cylinders are connected in series, the first producing a low vacuum and the second a high vacuum. The two rotors are mounted on the same shaft, and are rotated by an electro-motor. The combination is immersed in a cast iron box filled with oil to prevent air leakage into the high vacuum. Such a pump reduces the pressure to 10^{-3} mm. (from the atmospheric pressure) in a few minutes.

53. The Diffusion Pump.—The *mercury vapour pump* or the *diffusion pump*, which has displaced all other pumps for producing high vacuum, was first devised by Gaede in 1915. Several patterns have since been developed on the same principle: we shall, however, consider here, on

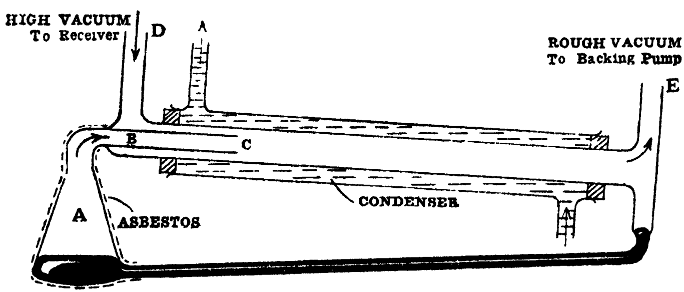

Fig. 52 Waran Pump, B—model.

account of its simplicity combined with efficiency, the **Waran pump**, designed by Dr. H. Parameswaran, India.

The apparatus (Fig. 52, B-model) with the receiver connected to D undergoes a first stage of evacuation by means of an oil pump (the backing pump), such as the hyvac pump, connected to E. Mercury is then boiled in the vessel A, and its vapour passes through the parallel jet tube BC (about 8 mm. in diameter). At C the gas from the receiver diffuses into the mercury vapour and is carried forward by it. The gas is finally removed by the backing pump, but the mercury vapour is condensed by a cold water condenser. The condensed mercury is arranged to flow back into the boiler. The condenser also prevents the flow of mercury vapour into the receiver. This kind of pump is also known as the **Condensation pump.**

The Waran pump will rapidly produce the highest vacuum (about 10^{-8} mm.) with a backing pressure less than 0·5 mm.

The Steel Bottle Pump.— In a glass pump there is always the risk of breakage. Hence steel pumps are coming into use, especially in industry. Fig. **53** indicates a very simple design of mercury vapour pump in steel, called the **steel bottle pump**. This requires a forevacuum of about $\frac{1}{100}$ mm. The principle is the same as in Waran's pump.

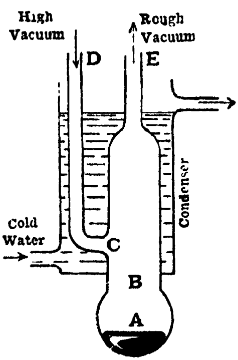

Fig. 53
The steel bottle pump.

Recently it has been found that a heavy mineral oil, named **Apiezon,** can be used in place of mercury in the diffusion pumps.

Pressure Gauges

54. The Open U-tube Manometer.—A *manometer* or *pressure gauge* is an instrument used for measuring fluid pressure. The open U-tube manometer is used for measuring pressures not much greater or smaller than the atmospheric pressure. It consists of a glass tube bent into the

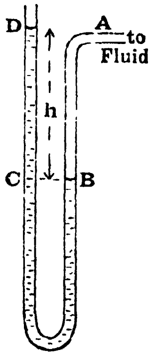

Fig. 54a.—U-tube manometer. Press. greater than 1 atmosphere.

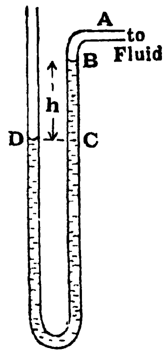

Fig. 54b.—U-tube manometer. Press. less than 1 atmosphere.

form of a U, as in Figs. 54a and 54b, and open at both ends. The tube is filled with a liquid, usually water or mercury, whose density (d) is known, until it occupies about half of each limb. The instrument is fixed vertically and the fluid (liquid or gas) whose pressure is to be measured is put in communication with A by means of a rubber tube. Then the difference in level (h) of the liquid surfaces B and D in the two limbs is noted with a scale.

Let us suppose for the sake of convenience in description that the pressure to be measured is due to a gas.

Let P be the atmospheric pressure in grams weight per sq. cm.

The pressure of the gas to be measured may be greater, as in Fig. 54a, or less, as in Fig. 54b, than the atmospheric pressure. In the first case the pressure of the gas acting on B = pressure on C at the same level = atmospheric pressure + pressure due to the column of liquid h (*i.e.*, CD) $= P + hd$ gm. weight per sq. cm. (where d is the density of the liquid). In the second case, the pressure of the gas on B + pressure due to column of liquid h (*i.e.*, CB) = atmospheric pressure. Hence the pressure of the gas $= P - hd$ gm. weight per sq. cm.

55. The Compressed-air Manometer.—This instrument (Fig. 55), used for measuring high pressures, consists generally of a U-tube closed at one end and open at the other and fixed vertically. The closed limb must be uniform. Mercury is poured into the tube, enclosing a column of air in the closed limb. The length of the air column is noted and its pressure is found from the barometer reading and the difference of levels. The open end is then connected with the fluid whose pressure is to be determined, and the length of the air column and the difference of levels again noted. By applying Boyle's law the pressure of the enclosed air is calculated from its original and present volumes and its original pressure. From this

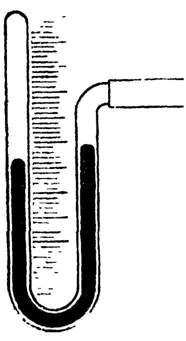

Fig. 55,
The compressed-air manometer.

the pressure of the fluid is found by adding or subtracting the difference of level in the second case.

56. The U-tube Vacuum Gauge.—For measuring low pressures, such as the pressure of air left in the receiver of a piston type air pump after exhaustion, the U-tube vacuum gauge is employed. It consists of a U-tube closed at one end and open at the other (Fig. 56). The whole of the closed limb and a portion of the other contain mercury. Let the pressure in the receiver of an exhaust pump be required. Then the open end is put in communication with the receiver. As the pump is being worked, at a certain stage the mercury in the closed limb will begin to fall, forming a Torricellian vacuum above. The pressure at any subsequent stage is given by the height of the mercury surface in the closed limb above that in the open limb. If a practically perfect vacuum be reached, the two levels would be equal.]

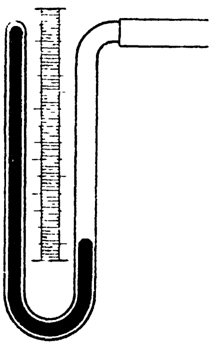

Fig. 56.
The U tube vacuum gauge.

57. The McLeod Vacuum Gauge.—When the pressure to be measured is less than a millimetre of mercury, the U-tube vacuum gauge cannot be used, as it is impossible to measure such a small difference of level with accuracy. In such cases the McLeod vacuum gauge is used. The principle of this gauge is that a large volume of the gas at the low pressure is compressed into a small volume, thereby increasing the pressure several times (maximum, 100,000). This increase of pressure is measured with a fair degree of accuracy.

The McLeod gauge (Fig. 57) consists of a narrow uniform tube A attached to the top of a cylindrical bulb B. The tube CE, branching at C just below the bulb B, is connected to the vessel, the pressure in which is to be measured. The lower end of tube CD is connected by means of rubber tubing to a reservoir R containing mercury. The total volume V of the bulb B and the narrow tube A is known by previous measurement (from the mass of water, for example, contained in them). The volumes of the narrow tube A between the graduations on the scale are also known by previous calibration (mercury thread method). To eliminate capillarity effect, the tubes A and E should have the same internal diameter.

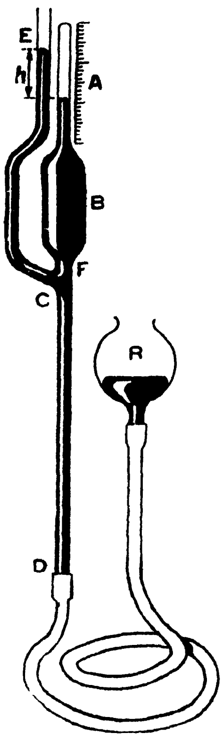

Fig. 57.
The McLeod vacuum gauge

The reservoir is raised slowly, until BA is just sealed off by the mercury level reaching F. The product of the pressure and volume of the gas in BA is then pV, where p is the low pressure to be measured. The reservoir is then raised further until the mercury occupies the narrow tube A, so that the difference in level h of the mercury surfaces in the tubes A and E can be measured with accuracy. The pressure of the gas in the system connected to E is practically unaltered by the change, since the additional volume occupied by the mercury in CE is small compared with the volume of the system connected to E. Now let v be the volume of the enclosed gas in A. The pressure of the gas

enclosed in A is $p+h$ and the product of its pressure and volume is thus $(p+h)v$. Hence by Boyle's law,

$$pV=(p+h)v. \quad \text{Hence}$$

$$p = hv / (V - v).$$

[In the denominator, v is usually negligible in comparison with V.]

The McLeod gauge is an *absolute gauge*. It can be used for measuring pressures down to 10^{-4} mm. When the pressure is much lower than this, the difference in level h becomes too small to be read with accuracy. The introduction of mercury vapour into the receiver system from this gauge can be prevented by means of the liquid air trap. The gauge is connected to the receiver before exhaustion commences.

58. The Decrement Gauge.—Besides the McLeod gauge, there are other gauges, such as the *radiometer gauge*, the *viscosity gauge*, the *hot-wire gauge* and the *ionisation gauge*, which are employed for measuring very low pressures. We shall consider here only one of these, namely, the *decrement gauge* (based on viscosity), being the most appropriate in this book.

At moderate pressures, the viscosity of a gas is independant of the pressure. At low pressures, however, the viscosity is a function of the pressure, and hence a body oscillating in a space containing gas at low pressure undergoes damping to an extent depending upon the pressure of the gas. The determination of the *logarithmic decrement* of an oscillating body can, therefore, be made the basis for measuring low pressures.

It can be shown that the logarithmic decrement λ of a body oscillating in a gas at low pressure is directly propor-

tional to the pressure p and to the square root of the molecular weight M, the temperature being constant.

$\lambda = k\,p\,\sqrt{M}$ for a single gas, or

$\lambda = k\,(p_1\sqrt{M_1} + p_2\sqrt{M_2})$ for a mixture of two gases, where k is a constant and p_1 and p_2 are the partial pressures.

King devised a type of decrement gauge in which the oscillations are photographed to facilitate the determination of λ. A short quartz fibre R (Fig. 58), each end of which is fused into a small sphere, is fixed horizontally at the middle to another quartz fibre mounted vertically under tension in a silica frame-work. The tension is thus practically unaffected by changes of temperature. A minute piece of iron at the junction of the fibres serves to start the oscillations of the horizontal fibre, the damping of which is determined. Light from a bright source is reflected on the suitable fused sphere of the vibrating fibre towards a moving or virtually moving photographic plate (cf. the falling plate experiment).

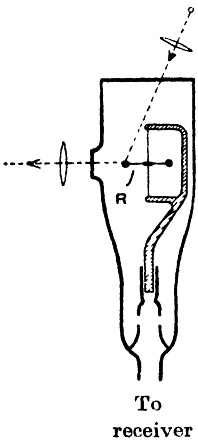

Fig. 58. King's decrement gauge.

The range of the instrument is from 10^{-2} to 10^{-6} mm. pressure. By calibrating the instrument by means of a McLeod gauge for the higher pressures, lower pressures can also be found.

[*Logarithmic Decrement*.—Let a_1, a_2, a_3......a_n be the successive amplitudes of an oscillating body. Then it can be proved that $a_1 / a_2 = a_2 / a_3 = a_3 / a_4 = \ldots = a_{n-1}/a_n =$

a constant, say, C. Then $\log_e C$ is called the *logarithmic decrement,* usually represented by λ.

Multiplying together the ratios,

$$a_1 / a_n = C^{n-1}$$

$$\therefore \quad (n-1)\log_e C = \log_e a_1 - \log_e a_n$$

$$i.\,e.,\ \lambda = (\log_e a_1 - \log_e a_n)/(n-1).$$

Thus λ can be determined.]

EXAMPLES

1. The barrel of a condensing air-pump is one inch in diameter and 8 in. long. The tube of a pneumatic tyre when inflated is one inch in diameter and 80 in. long. If, to begin with, the tyre is empty, how many strokes of the pump will be needed to inflate it with air at twice the atmospheric pressure ?

Volume of inflated tyre : vol. of barrel = 10 : 1.

Let n be the number of strokes required.

Then $n\,P = 10 \times 2\,P$, where P is the atm. pr.

Hence $n = 20$.

2. The length of the air column in a compressed-air manometer is 20 cm., when the level of mercury in the closed limb is 5 cm. below the level in the other and the atmospheric pressure is 76 cm. The open end is then connected to a gas cylinder, when the length of the air column is found to be 10 cm. and the level in the closed limb to be 15 cm. above the level in the other. Find the pressure of the gas in the cylinder.

The original pressure p_1 of the air in the manometer = $76 + 5 = 81$ cm.

Let the pressure of the air after connecting be p_2.

Then, since $p_1 v_1 = p_2 v_2$, $81 \times 20 = p_2 \times 10$, i.e., $p_2 = 162$ cm.

$\therefore$ the pressure of the gas in the cylinder $= 162 + 15 = 177$ cm. of mercury.

3. In a McLeod gauge, $V = 500$ c. c. In measuring the pressure in an exhausted vessel, $h = 5$ mm. and $v = 0{\cdot}02$ c. c. Calculate the pressure.

$$p = hv / (V - v) = 5 \times 0{\cdot}02 / (500 - 0{\cdot}02) = 0{\cdot}0002 \text{ mm.}$$

QUESTIONS ON CHAPTER V

(1) Describe the construction and working of a modern high vacuum pump.

How would you measure such low pressures?

[M. U., B. Sc., Sep. 1937]

(2) Describe and explain the action of a pump which produces pressures of the order of 10^{-4} mm.

Explain a method of measuring the pressures attained.

[M. U., B. Sc., Sep. 1938]

(3) Describe a form of a modern pump designed to produce a vacuum of the order of $0{\cdot}01$ mm. and explain its action.

Show how the low pressure obtained may be measured.

[M. U., B. Sc., March 1939]

CHAPTER VI

SURFACE TENSION

59. The Free Surface of a Liquid behaves like a Stretched Elastic Membrane.— The following phenomena show that the free surface of a liquid behaves like a stretched elastic membrane :—

(1) Place a new (hence slightly oiled), thin, steel needle on a small sheet of blotting paper and float the latter on water. *Gently* submerge the paper under the water without disturbing the needle. The needle will now be found floating on the water, in spite of its higher density. The weight of the needle here produces a slight depression on the water surface. If the needle is cleaned free of oily matter, the water wets the needle, which consequently sinks.

That certain insects, such as mosquitoes, can float and move on the surface of water without sinking is explained in a similar manner. Contamination of the water surface with kerosene lowers the strength of the surface film, which consequently becomes unable to support the insects.

(2) Make a plane ring of wire and, by dipping it into soap solution and taking it out, get a plane film on it. Place on the film a closed loop of thin cotton thread. The loop can be made to take any form such as A (Fig. 59). Now pierce the film inside the loop. The loop immediately takes up the circular form B ; if it is deformed in any way, it springs back to the circular form on being released.

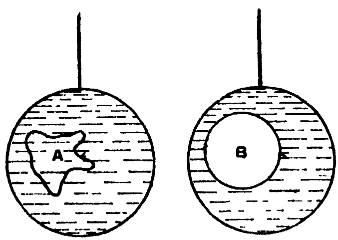

Fig. 59.

The circle is the plane figure, which, for a given perimeter, has the maximum area. Hence the above experiment demonstrates that the *film* tends to occupy the least possible surface, just as a stretched elastic membrane would do.

Again tie a piece of thin cotton thread loosely across the ring. As before get a film of soap solution on it. Then pierce one portion of the film. The thread will now be pulled out in the form of an arc of a circle (Fig. 60).

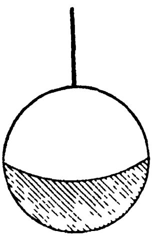

Fig. 60.

(3) If an elastic uniform bag, such as a toy rubber balloon, is inflated with air, it always takes the shape of a sphere. Here the air tries to expand and the elastic membrane tends to contract, so that the shape assumed must be the one in which a given volume is enclosed within the minimum area, namely, the sphere. It has been found that liquid drops, when not deformed by gravity, are always spherical in shape. The nature of the rainbow affords proof that the drops of water, which give rise to it, must be perfectly spherical. (Since the drops fall freely, their shape is practically not influenced by gravity).

Again, an oil dropped into a mixture of water and alcohol, having the same density as the oil, assumes a perfectly spherical shape within the mixture.

Lead shot is manufactured by pouring down molten lead in a fine stream from the top of a high tower. The stream breaks up into a series of small spherical drops, which solidify as they fall through the air. (They are caught in a deep bath of water to prevent being knocked out of shape when they reach the ground.)

A big drop of mercury on a glass plate has the same flattened shape as a thin elastic balloon filled with water and placed on the table. Here the flattening is due to gravity.

(4) If a (non-oily) paint-brush is dipped into water, the hairs of the brush will be seen to lie apart, but on raising the brush out of the water, the hairs cling together, showing that the water surface tends to shrink.

(5) Spread a thin layer of water on a clean glass plate and place a drop of ether or oil upon it. The water will be found to retract from the drop. Here the tension of the water surface is greater than the tension of the ether or oil surface, which explains the pull.

Float a paper boat on water and attach a piece of camphor to the stern of the boat in contact with water. The boat will now move forward, the pull of the water surface being greater than the pull of the camphor-water surface formed by camphor dissolving a little in water.

There is one important difference between a liquid film and a stretched elastic membrane; while the tension in the latter increases with the amount of stretching, the tension in a liquid film is independent of the stretching

60. Explanation for the Tension in a Liquid Surface.— Each molecule in a liquid is attracted by the other molecules, but when the distance between two molecules exceeds a certain value, called the *range of molecular attraction*, the attraction becomes negligible. Hence if we

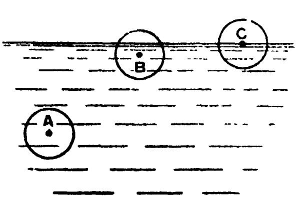

Fig. 61

wish to consider the effect of attraction on a molecule A

(Fig. 61) by the neighbouring molecules, we have only to consider the attraction exerted on A by the molecules lying within the sphere constructed with A as centre and the range of molecular attraction as the radius.

In the case of a molecule (A) lying well within the liquid, the whole sphere lies within the liquid and hence the molecule will be attracted equally in all directions. If, however, the molecule (B) is so situated that the sphere intersects the surface of the liquid, it is clear that the downward force on the molecule is greater than the upward force, and hence there is a resultant force downwards. In the case of a molecule (C) lying on the surface itself, this resultant force on the surface is a maximum. On account of these unbalanced forces acting on the molecules on or very near the surface, the surface film possesses a compact structure with a comparatively high cohesive force per unit area. Thus the surface film possesses the property of a mobile and somewhat tenacious sheet. As the range of molecular attraction (exaggerated in Fig. 61) is small, the thickness of the surface film exhibiting this property is also small.

On account of this inward attraction exerted on a molecule when it approaches the surface, work must be done to transfer a molecule from the interior of a liquid to the surface. Thus the free surface of a liquid possesses a definite amount of potential energy per unit area, under the same physical conditions. Since a system tends to adjust itself so as to possess the minimum potential energy possible, a liquid always tends to adjust itself so as to possess the minimum area of surface, making due allowance for the associated gravitational potential energy. Thus a liquid surface always tends to contract in area, and this tendency endows it with the property of a stretched elastic membrane.

61. Definition of Surface Tension.—Suppose we have a liquid film held in the light framework ABCD (Fig. 62) of which the sides AB, AD and BC are fixed, while CD is movable. Then CD will be pulled up owing to the surface tension, and hence, in order to keep it in equilibrium, a force F must be applied to it in the opposite direction (perpendicular to DC) as in the figure. The length of the film pulling up DC is 2DC, as the film exists on both surfaces of the rectangle. The quantity $F/2DC$ is defined as the **surface tension** of the liquid. *It is the force exerted by any edge of the film per unit length, and is expressed in dynes per centimetre* in the C. G. S. system.

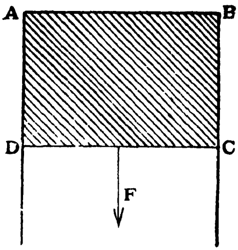

Fig. 62.

The surface tension of water is about 70 dynes per cm. and of mercury about 480 dynes per cm. at 30°C. Surface tension decreases as the temperature increases.

62. Surface Energy in Liquid Film.—Consider a liquid film held in the light rectangular frame ABCD (Fig. 63), of which CD alone can slide along AD′ and BC′, the other sides being fixed. Let CD be pulled to the position C′D′ through a distance x. The force applied here is $2l.\ T$, where l is the length of CD and T the surface tension of the liquid, and hence the work done in pulling CD through x is $2\ l\ T.\ x$. The area of the film newly created is $2\ l\ x$ (for both surfaces), and hence the *mechanical work* done to produce unit area of the film is numerically equal to $2\ l T x\ /\ 2 l\ x$ or T.

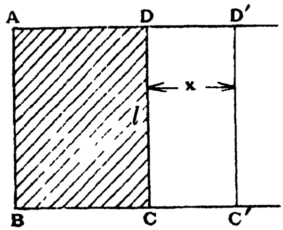

Fig. 63.

While the film is being stretched, it cools and hence absorbs heat from the surroundings in attaining the common temperature. Hence the potential energy or surface energy per unit area of the film consists of both the mechanical energy spent and the heat energy absorbed by it in its formation. Hence *the mechanical energy required to produce unit area of a liquid surface is numerically equal to its surface tension.*

63. Surface Tension is the same (1) at All Points and (2) in All Directions—(1) Consider a very narrow rectangle ABCD (Fig. 64) of infinitesimal breadth dl on the surface of a liquid. Forces act on the edges normally, since a liquid cannot withstand a shearing force.

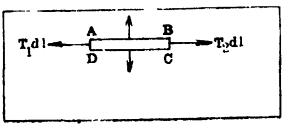

Fig. 64.

Let T_1 and T_2 be the surface tensions at A and B respectively. Then resolving the forces along the line AB, we get for equilibrium

$$T_1\, dl = T_2\, dl.$$

$$\text{Hence } T_1 = T_2.$$

Since A and B are any two points, it follows that *the surface tension is the same at all points.*

(2) Consider the equilibrium of an *element*, in the surface of a liquid, in the shape of a right-angled triangle ABC (shown magnified in Fig. 65).

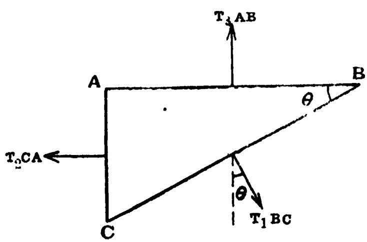

Fig. 65.

Let the surface tensions at right angles to the edges BC, CA and AB be T_1, T_2 and T_3 respectively. Then the forces acting on the edges BC, CA and AB are T_1 BC, T_2 CA and T_3 AB respectively. Resolving the

forces in the line at right angles to AB, we get, for equilibrium,

$$T_3\,AB = T_1 .\ BC \cos ABC = T_1 .\ AB.$$

$$\therefore\quad T_3 = T_1.$$

Since angle ABC ($=\theta$) may have any value, it follows that *the surface tension is the same in all directions.*

64. Why some Drops are Spherical and some Flattened.—*When a mechanical system is in equilibrium, the total potential energy in the system must be a minimum.* The total potential energy (P.E.) of a drop consists of (1) the P.E. due to gravity and (2) the P.E. due to surface tension. Take the case of oil dropped into a mixture of water and alcohol, of the same density (Art. 59). Here change in the shape of the mass of the oil will not affect the P. E. of the system due to gravity, since the density of the oil is the same as that of the surrounding liquid. The only change in the P.E. must, therefore, be due to surface tension. The oil will therefore assume the shape in which the P.E. due to surface tension is a minimum, *i. e.*, the shape in which the area is a minimum. Now the sphere has the smallest surface enclosing a given volume. Hence the drops of oil in the liquid are spherical.

Again, consider a drop of mercury on a glass plate. If the drop is large, the potential energy due to gravity becomes predominant, while the potential energy due to surface tension is small; if the drop is small, the former becomes insignificant, while the latter becomes predominant (p. 5). Hence a *large* drop becomes flattened so as to lower the P.E. due to gravity, even though this involves a slight increase in the P.E. due to surface tension; and a *small* drop takes very nearly the spherical shape, as the P.E. due to surface tension will then be a minimum.

65. Relation between the Radius of a Spherical Drop of Liquid, the surface Tension and the Pressure.— Imagine the spherical drop of the liquid to be cut into two hemispheres, and consider the equilibrium of one of them ABCDE (Fig. 66). If there were no surface tension, a uniform pressure on the plane surface ABCD would balance an equal uniform pressure on the curved surface AEC (cf. Art. 21). As it is, however, the surface film exerts its own pressure inwards, due to tension, so that the pressure inside will have to be greater than the pressure outside the sphere for equilibrium. Let p be the excess of internal pressure over the external. Let r be the radius of the sphere and T the surface tension of the liquid. The only forces that must be considered here (see Art. 64) for equilibrium are (1) the thrust on the plane face ABCD exerted by the other half, due to the excess of internal pressure over the external and (2) the pull on the edge of the circle ABCD exerted by the edge of the other half due to surface tension. These forces must be equal and opposite. Hence

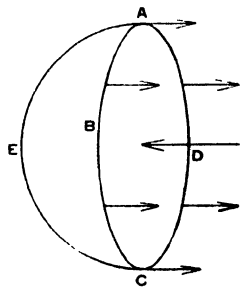

Fig. 66.

$$p \times \pi r^2 = T \times 2 \pi r.$$
$$\therefore \quad p = 2 T / r.$$

This applies to a *drop*, which has only *one* surface film. In the case of a *soap bubble*, in which there are *two* surface films, one inner and the other outer,

$$p \times \pi r^2 = T \times 4 \pi r. \quad \text{Hence}$$
$$p = 4 T / r.$$

It must be noted here that the excess of pressure is *inversely* proportional to the radius. The pressure inside a

smaller soap bubble is therefore greater than the pressure inside a larger one. This can be demonstrated as follows: Dip the ends D and E of the apparatus represented in Fig. 67 in soap solution, and then take them out. Open C and A, close B and blow a small bubble at D. Then close A, open B and blow a large bubble at E. Now close C and open A. The small bubble (at D) will now be found to become smaller and smaller and the large bubble (at E) larger and larger, showing that air flows from the small bubble to the large one.

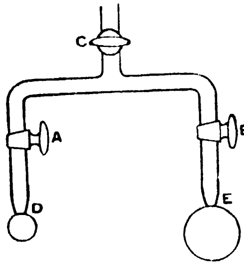

Fig 67.

66. Relation between the Radius of a Cylinder of liquid, the Surface Tension and the Pressure. — Imagine a portion (Fig. 68) contained between two planes at right angles to the axis of the cylinder. Divide the portion into two halves by a plane BEFC passing through the axis of the cylinder. Now consider the equilibrium of the half ABCFDE.

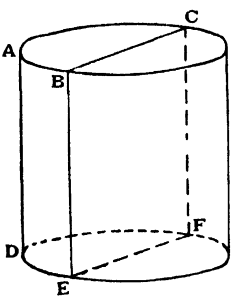

Fig. 68.

Let r be the radius of the cylinder, T the surface tension, p the excess of pressure inside over that outside the cylinder, and h the length BE.

The forces that must be considered here are (1) the thrust on the face BEFC equal to $p \times 2rh$, (2) the forces (of pull) on the edges BE and CF due to surface tension, each of which is equal to Th, (3) forces on the edges ABC, DEF and

thrusts on the faces ABC, DEF, which are all parallel to the axis of the cylinder.

Resolving the forces along the line perpendicular to the face BEFC, we get

$$p \times 2rh = 2\,T\,h. \quad \text{Hence}$$

$$p = T\,/\,r.$$

67. Relation between Surface Tension, Pressure and Radii of Curvature for Any Surface.—Let us now consider a liquid surface which has its *principal radii of curvature* equal to r_1 and r_2 respectively. Let AA′ and BB′ be the *principal sections.* Take an *element of* the surface bounded by sides symmetrically cut by planes parallel to the principal planes (Fig. 69). Let AA′ $=a$ and BB′ $=b$ in the element (magnified in the figure). Then considering the principal section AA′ (Fig. 70), the forces of pull due to surface tension on the sides passing through A and A′ are each equal to bT, acting tangentially. (The sides are so small that the variation in the direction of T is negligible.) Hence their resultant is equal to $2bT\cos\frac{1}{2}\,AO_1A' = 2\,bT\sin\frac{1}{2}\,\theta = 2bT.\ \frac{1}{2}\,\theta = bT\,\theta$, since θ

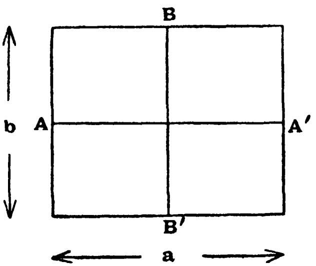

Fig. 69.

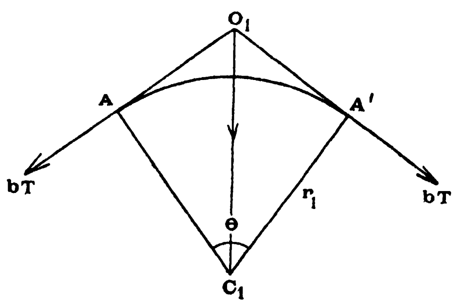

Fig. 70.

is small. Hence the resultant $= bTa/r_1$. This acts in the direction $O_1 C_1$. Similarly, for the other principal section B B′ the resultant $= a T' b/r_2$. Hence the total resultant $= abT/r_1 + abT/r_2$, if the two curvatures are *similar.* Let an excess of internal pressure, equal to p, over the external be required to balance this. Then the thrust due to this is abp. Hence for equilibrium,

$$abp = ab\,T/r_1 + ab\,T/r_2\,,\ i.e.,$$

$$p = T/r_1 + T/r_2 \ldots\ldots\ldots\ldots \quad (1)$$

This being a general equation, we can apply it to the sphere and the cylinder. In the sphere, $r_1 = r_2 = r$, say, and hence $p = 2\,T/r$. In the cylinder, one radius $= \infty$, and hence $p = T/r$.

The equation (1) derived above applies to a surface in which the two centres of curvature are on the same side, as a spherical or ellipsoidal surface. Such a surface is called a *synclastic surface.* For a surface in which the two centres of curvature are on opposite sides, the equation should be

$$p = T/r_1 - T/r_2 \ldots\ldots\ldots\ldots \quad (2)$$

Such a surface is called an *anticlastic surface.* The surfaec of a saddle is an example of this. Another example is the surface of the film in Fig. 71.

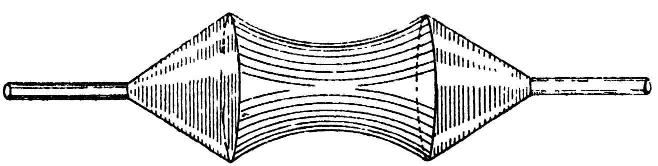

Fig. 71.

If, however, we adopt the convention that radii of curvature of opposite curvatures (anticlastic surface) have opposite signs, equation (2) merges in equation (1).

The film formed between the rims of two open funnels has an anticlastic surface. Here since the pressures inside and outside are equal,

$$T/r_1 + T/r_2 = 0.$$

Hence the numerical values of r_1 and r_2 are equal.

68. Force between two Plates held together by Liquid Film.—If a drop of water be squeezed between two clean plates of glass, a considerable force will be required to pull the plates apart. This is due to the effect of surface tension.

Let P and Q be two clean, plane plates (say, of glass), between which a drop of a liquid which wets them (say, water) is squeezed. Let d be the distance between the inner surfaces of the plates and D the diameter of the liquid disc between the plates. Then the radii of curvature of the free surface of the liquid are $d/2$ and $-D/2$. Hence the pressure outside (*i.e.* the atmospheric pressure) is greater than the pressure inside the liquid by $2T\ (1/d - 1/D)$. Here d is very small compared with D, hence the difference of pressure is very approximately equal to $2\ T/d$. If A is the area of each plate wet by the liquid, the force F urging P towards Q is given by

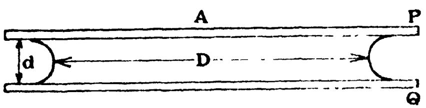

Fig. 72.

$$F = 2\ AT\ /d.$$

The force thus varies inversely as the distance between the plates. If a drop of water is placed between two plates of glass, the plates are forced together, which still further increases the force between the plates as the area of the wet surface increases and the distance between the plates diminishes.

69. Spreading of one Liquid on Another.—If a drop of one liquid B were formed on the surface of another liquid C (Fig. 73), there would be three surface tensions to be considered for equilibrium: (1) T_{ba} between liquid B and air (2) T_{ca} between liquid C and air (3) T_{bc} between liquids B and C.

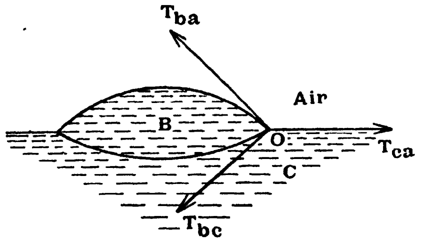

Fig. 73.

For the equilibrium of a small length dl at O taken perpendicular to the plane of the figure, the three forces ($T_{ba}dl$, $T_{ca}dl$ and $T_{bc}dl$), according to the law of the triangle of forces, must be such that it should be possible to represent them by the sides of a triangle. This means that the sum of *any* two surface tensions must be greater than the third. There are no two *pure* liquids known, for which this is possible, one of the three surface tensions being always greater than the sum of the other two. The lighter liquid thus always *spreads* over the surface of the liquid on which it rests, without forming a drop. Thus a drop of pure water, placed on the surface of pure mercury, spreads over it, forming a uniform layer. If however, the surface of mercury is contaminated with oily matter, it is possible for a drop of water to stand upon it without spreading. Here the surface tension of the mercury is lowered on account of the impurity, and it is then possible to construct the triangle.

The triangle of forces referred to above is, in this connection, called *Neumann's triangle*.

70. Liquid on Plate forming a Drop.— We must regard the surface of a solid also as possessing surface tension,

as in the case of a liquid surface, though the former is not mobile. The process of solidification must increase, rather than decrease, the surface tension, as lowering the temperature always increases the surface tension.

Now consider a liquid drop L on a plane solid surface S

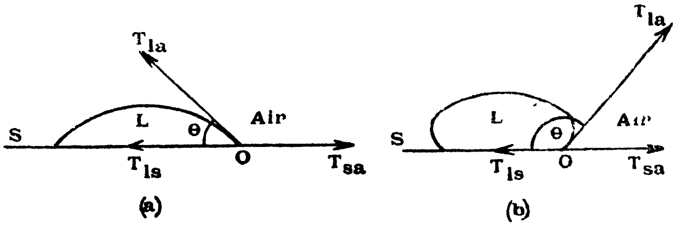

Fig. 74.

(Fig. 74). Let θ be the angle at which the liquid meets the solid surface, *the angle being measured inside the liquid.* The forces acting on an element of length dz at O, taken perpendicular to the plane of the figure, are $T_{la}\,dz$, $T_{sa}\,dz$ and $T_{ls}\,dz$, in the directions indicated in the figure. Resolving the forces along the line of the solid surface S and cancelling the common dz, we get for equilibrium

$$T_{ls} + T_{la}\cos\theta = T_{sa} \qquad \ldots\ldots\ldots\ldots(1)$$

$$\therefore \quad \cos\theta = (T_{sa} - T_{ls})/T_{la} \qquad \ldots\ldots\ldots\ldots(2)$$

If $T_{sa} > T_{ls}$, cos θ is positive, and θ is less than 90°. If $T_{sa} < T_{ls}$, cos θ is negative, and θ lies between 90° and 180°.

If, however, $T_{sa} > (T_{ls} + T_{la})$ (*i.e.*, $T_{sa} - T_{ls} > T_{la}$), there cannot be equlibrium (from equation (1), or from equation (2) as cos θ must always be less than 1), and the liquid will then *spread* over the solid.

71. Angle of Contact.— In Art. 70 we derived that $\cos\theta = (T_{sa} - T_{ls})/T_{la}$ (Fig. 74). Thus θ is a constant for

the same three substances. This angle θ is called the *angle of contact* between the liquid and the solid in air. *The angle of contact is* thus *the angle between the tangent to the liquid surface at the point where it meets the solid and the surface of the solid, the angle being measured inside the liquid.* The angle of contact also depends, of course, on the third material, which, when not otherwise stated, is always taken to be air.

If the liquid wets the solid, such as water and glass, the angle of contact is nearly 0°. For mercury and glass (in air) θ is about 140°.

72. Experimental Determination of Angle of Contact.— (i) Take a clean, dry flask with spherical bulb about 5 cm. in diameter. Pour into it clean mercury until it is nearly full, and then close the mouth of the flask with a rubber stopper into which a thick piece of glass rod has been inserted. Invert the flask and adjust the rod so that the mercury surface is plane *right up to the glass* (Fig. 75). This can be tested by observing the image of an object (such as printed matter) reflected on the liquid surface at the line of contact. Even the slightest curvature will then be indicated by the distortion in the image. [At a lower level than this the mercury surface at the line of contact would be convex, and at a higher level, concave.]

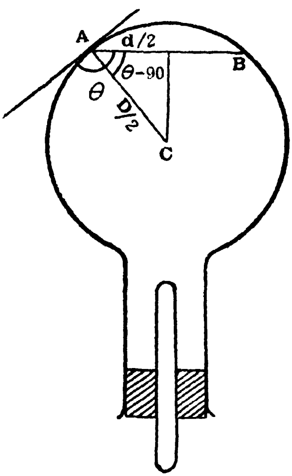

Fig. 75

Measure the diameter d $(=AB)$ of the plane mercury surface with calipers and the diameter D $(=2\ AC)$ of the spherical bulb. The required angle of contact θ can then be obtained from.

$$\cos(\theta - 90^\circ) = \tfrac{1}{2}d/\tfrac{1}{2}D = d/D.$$

(ii) Dip a clean, plane glass plate into mercury contained in a trough, and incline it slowly until the under surface CB (Fig. 76) is plane *right up to the plate* [tested by absence of distortion in reflected image at the line of contact, as in (i)]. Clamp the plate in this exact position, and with a plumb-line measure AB and BC. Then

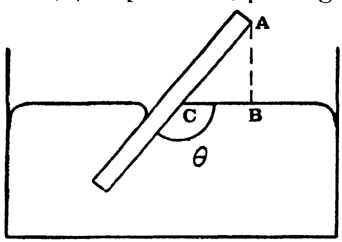

Fig. 76.

$$\tan(180^\circ - \theta) = AB/BC,$$

where θ is the angle of contact.

73. Capillarity.—If a clean glass tube of *fine* bore (called a *capillary* tube) is dipped into water, the water rises in the tube and stands at a higher level. This phenomenon is called *capillarity*, and is due to surface tension. The ascent of oil in a wick, the rise of sap in plants and trees, the spread of water through a lump of sugar and the absorption of ink by blotting paper are other familiar examples.

The capillary rise of liquid (Fig. 77) takes place only when the angle of contact is less than 90°. In the case of liquids which wet glass, such as water, the angle of contact may be taken as zero.

But in the case of liquids, such as mercury, whose angle of contact with glass is greater than 90°, there is a capillary *depression* (Fig. 78).

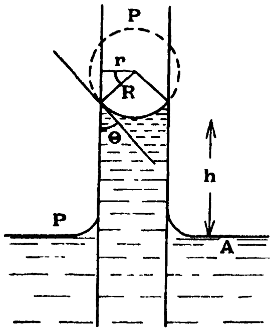

Fig. 77

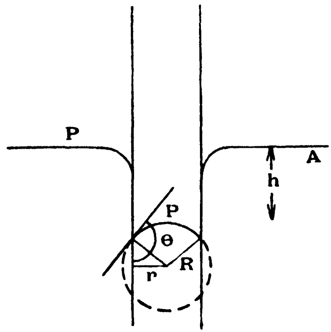

Fig. 78.

We shall now investigate the relation between the extent of capillary elevation or depression h and the other quantities involved. In a narrow tube of circular cross-section, the surface of the liquid is nearly spherical. Let the radius of this spherical surface be R. Let the radius of the bore of the tube be r, the density of the liquid ρ, and the angle of contact θ.

If P is the common pressure above the liquid inside and outside the tube, the pressure just below the concave surface (Fig. 77) is $P-2T/R$, and the pressure just below the convex surface (Fig. 78) is $P + 2T/R$ (Art. 65). The pressure due to the column of liquid of height h is $h\rho g$. Since the pressure at the flat surface A of the liquid is P,

$$P-2\,T/R + h\rho g = P$$

in the case of capillary elevation, and

$$P + 2T / R = P + h \rho g$$

in the case of capillary depression. Thus in each case,

$$2T / R = h \rho g.$$

But $r / R = \cos\theta$ in elevation, and $\cos(180-\theta)$ in depression. Hence

$$\frac{2\,T\cos\theta}{r} = h\,\rho\,g, \text{ where } h \text{ is the elevation,}$$

and $$\frac{2\,T\cos(180^\circ-\theta)}{r} = h\,\rho\,g, \text{ where } h \text{ is the depression.}$$

The first equation may be made to represent the second case also, if we adopt the convention that a negative value of h means depression.

74. Rise of Liquid between Parallel Plates.— Let d be the distance between the parallel plates, other symbols meaning the same as before. Take a horizontal length of l cm. of the liquid along the plates, and consider the equilibrium of the liquid column of this length and of height h. Its weight is $h\,l\,d\,\rho\,g$. It is supported by two forces, due to surface tension, each equal to $T\,l\cos\theta$ (Art. 70, $T_{sa} - T_{ls} = T_{la}\cos\theta$). Hence

$$2\,T\,l\cos\theta = h\,l\,d\,\rho\,g, \text{ i.e.,}$$
$$2\,T\cos\theta = h\,d\,\rho\,g.$$

The rise h is thus inversely proportional to d. (The equation can also be derived by the method of Art. 73).

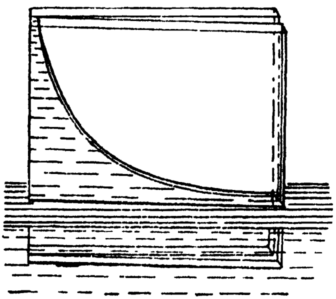

Fig. 79.

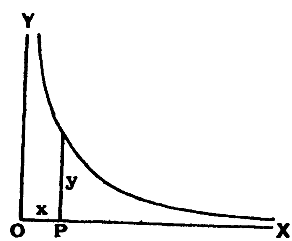

Fig. 80.

75. Rise of Liquid between two Inclined Vertical Plates. — Let two glass plates, inclined at a small angle and

in contact at one edge of each, be vertically dipped into a liquid (Fig. 79) which wets glass (say, water). According to the equation derived in Art. 74, the rise y at any point P (Fig. 80) will be inversely proportional to the distance d between the plates at that point. But this distance d is directly proportional to the distance x of P from O. Therefore $y \propto \frac{1}{x}$ or yx is constant. Therefore the curve formed by the top of the liquid which has risen is a rectangular hyperbola.

76. Attraction or Repulsion between bodies Partly Immersed in Liquid—In the case of two parallel plates

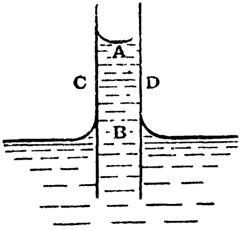

Fig. 81.

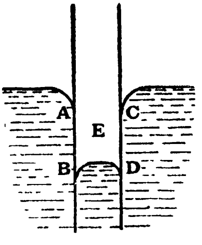

Fig. 82.

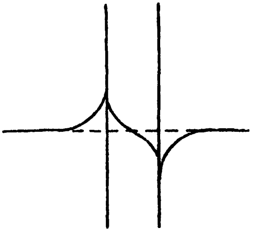

Fig. 83.

dipped in a liquid which wets them, the pressure at A (Fig. 81) is less than the external pressure above the meniscus by T/R. Below A the pressure goes on increasing until at B the pressure is equal to the external pressure. Hence the pressure in AB is less than the external pressure on the sides C and D. Therefore the plates tend to be forced towards each other.

If the liquid does not wet the plates (Fig. 82), it is clear that the pressures on the sides AB and CD are greater than the pressure at E, and therefore, the plates, tend to be forced towards each other in this case also.

If, however, one of the plates is wetted by the liquid, while the other is not, the surface of the liquid between the plates takes the form shown in Fig. 83 when the plates are very near each other. It will be seen that there is no horizontal portion in the surface there. Consideration of the forces due to surface tension shows that the plates tend to be forced *away* from each other in this case.

These results explain why bubbles on a liquid surface group together, and small sticks float together on water.

Determination of Surface Tension

77. Surface Tension by Capillary Rise Method. — Take a capillary glass tube and subject it thoroughly, inside and outside, to the action of a mixture of potassium bichromate solution and sulphuric acid to remove oily matter. Then wash it with tap water and finally rinse it with distilled water. Dry the tube if the liquid given is not water.

Attach a bent pin to the tube by bands, as in Fig. 84, dip the tube into the liquid contained in a beaker placed on a block, and clamp the tube vertically with the turned-up point of the pin L just on a level with the liquid surface. The portion above the meniscus in the tube should be wetted with the liquid, say, by dipping the tube a little deeper and then raising it to the proper position. By means of a travelling microscope measure the height h of the liquid

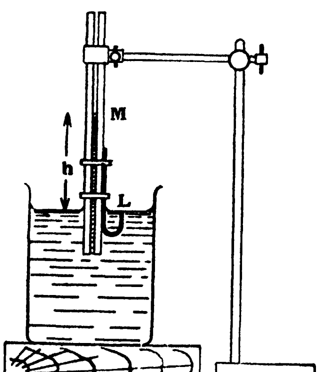

Fig. 84

column in the tube. For this, the reading corresponding to the meniscus M and, after removing the beaker, the reading corresponding to L are noted; the difference gives h. The position M corresponding to the meniscus must then be marked on the tube, and the radius r of the bore of the tube *at this position* must be found. If the bore of the tube is not uniform, the tube must be cut exactly at M and the average diameter of the bore at the section must be found by means of the travelling microscope. If the tube is fairly uniform, the radius may also be found either by introducing a short thread of mercury into the tube with its mid-point at M and measuring its length and mass, or by measuring the average diameter at the two ends and taking the mean value. The density of the liquid ρ and the angle of contact θ are then determined.

Applying the equation

$$\frac{2\,T\cos\theta}{r} = h\,\rho\,g$$

T is calculated. For liquids which wet glass (such as water) θ may be taken as zero, so that $T = r\,h\,\rho\,g\,/\,2$.

To allow for the liquid in the meniscus, $\frac{1}{3}\,r$ must be added to h in very accurate determinations.

78. Surface Tension by Balance Method.—Take a glass plate of the approximate size of a microscope slide and expose it to the action of a mixture of potassium bichromate solution and sulphuric acid to remove oily matter. Then wash it with tap water and finally rinse it with distilled water. Suspend it vertically by means of a clip attached to a longer edge, and with the corresponding lower edge

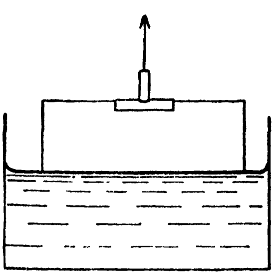

Fig. 85.

touching the surface of the given liquid horizontally (Fig. 85), find the force required just to pull the plate away from the liquid against the force of surface tension. This force (mg) may be found by means of a torsion balance, common balance or spring balance. The lower edge should not be soiled by touch.

Measure the length l and the thickness t of the edge which was in contact with the liquid. The total length of the film exerting surface tension is $2\ (l + t)$ and hence

$$mg = 2\ (l + t)\ T,$$

from which T is calculated.

79. Surface Tension of Soap Bubble.—The surface tension of a soap bubble can be conveniently determined by the arrangement shown in Fig. 86. The manometer AB contains a liquid of low density ρ. A small bubble is blown at D and the tap C is then closed. The diameter ($2r$) of the bubble and the difference in level (h) in the manometer are measured with a travelling microscope. Then

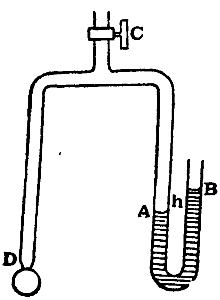

Fig. 86.

$$4T\ /\ r = h\ \rho\ g.$$

80. Surface Tension from Weight of Drop.—In this method a counted number of drops falling *very slowly* from a vertical tube is collected in a beaker and weighed (Fig. 87). The radius r of the lower end of the tube is then found with a travelling microscope. If the drops are confined to the bore by coating the rest of the section (lower end) with wax, the *internal* diameter should be measured. If the drops are

allowed to spread to the whole section (as in Fig. 87), without using wax, the *external* diameter should be measured.

The forces acting on each drop, when it is on the point of becoming detached, are (1) its weight mg acting downwards, (2) the force due to surface tension $2\pi rT$ acting upwards, (3) thrust, due to atmospheric pressure P, equal to $\pi r^2 P$ upwards, and (4) thrust, due to internal pressure P' in the plane of the orifice, equal to $\pi r^2 P'$ downwards. Hence

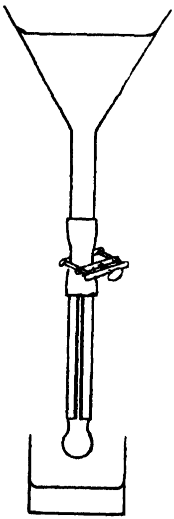

Fig. 87.

$$2\pi r T = mg + \pi r^2 P' - \pi r^2 P$$
$$= mg + \pi r^2 (P' - P).$$

But $P' - P = T / r$ (Art. 66).

$$\therefore 2\pi r T = mg + \pi r T. \quad \text{Hence}$$
$$T = mg / \pi r.$$

The above equation is far from being accurate, as the falling drop is not in statical equilibrium and the diameter of the cylindrical portion of the drop (taken to be $2r$ above) is an uncertain quantity. Lord Rayleigh finds the relation

$$T = mg / 3{\cdot}8\, r$$

to be sufficiently accurate for many purposes.

To determine the *interfacial tension* between two liquids, which do not mix, the lighter liquid is taken in a beaker and the lower end of the tube is dipped into this. The heavier liquid, poured into the tube, is allowed to drop slowly into the first liquid.

Owing to the force of buoyancy, we have to use here $mg\,(1-\rho_1/\rho_2)$ in place of mg, where ρ_1 and ρ_2 are the densities of the lighter and heavier liquids respectively.

[The volume of the drop is $m\,/\,\rho_2$ and the mass of an equal volume of the surrounding liquid is $m\,\rho_1/\rho_2$. Hence the apparent weight of the drop $=$ mg $-$ force of buoyancy $= mg - mg\,\rho_1\,/\,\rho_2$].

81. Surface Tension by Measurement of Drops and Bubbles : Quincke's Method.--Mercury is a suitable liquid for the method of determination of surface tension by measurement of drop. Fig. 88 represents a central vertical section of a *large* drop of mercury on a horizontal glass plate. The upper surface of the drop is practically horizontal except near the edge. At V and V′ the tangent planes to the surface are vertical.

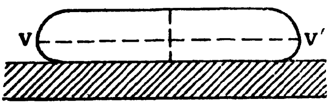

Fig. 88

Fig. 89 represents the drop as seen from above. Imagine the slice ABDCM (Fig. 90) formed by cutting the drop by

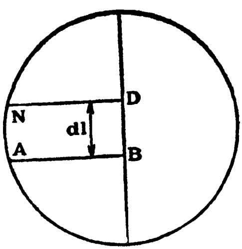

Fig. 89.

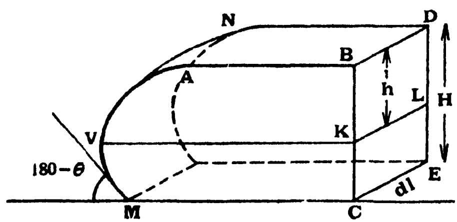

Fig. 90.

two parallel symmetrically placed vertical planes (passing through AB and ND) at distance dl apart and the central vertical plane (passing through BD) at right angles to these.

Let BK $= h$, BC $= H$, the density of the liquid $= \rho$, and its surface tension $= T$.

(i) Consider the equilibrium of the portion ABLKV that lies above the horizontal plane VKL. The only forces acting on it parallel to VK are (1) the pull on BD, by the other half, due to surface tension and (2) the thrust on the face BDLK due to the pressure exerted by the other half in excess of the atmospheric pressure. (The force due to surface tension at V is vertical, and the forces on ABKV and the corresponding one at the back are at right angles to VK.) Force (1) $= T\,dl$ and force (2) $=$ pressure at centre of surface $\times$ area $= \frac{1}{2} h \rho g \times h\,dl = \frac{1}{2} h^2 \rho g\,dl$. Hence

$$T dl = \tfrac{1}{2} h^2 \rho g\, dl.$$

$$\therefore \quad T = \tfrac{1}{2} h^2 \rho g \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

(ii) Consider the equilibrium of the whole slice ABECM and resolve the forces along the line VK. The forces to be considered here are (1) the pull on BD equal to Tdl, (2) the thrust on BDEC equal to $\frac{1}{2} H\rho g \times H dl$, i.e., $\frac{1}{2} H^2 \rho g\, dl$ and (3) the pull due to surface tension, in the direction MC, on the edge passing through M parallel to CE; this pull is equal to $Tdl \cos (180° - \theta)$, where θ is the angle of contact [Art. 70, equation (1)]. Hence

$$Tdl + Tdl \cos (180° - \theta) = \tfrac{1}{2} H^2 \rho g\, dl$$

$$\therefore \quad T = \frac{H^2 \rho g}{2\,[1 + \cos (180° - \theta)]} \ldots\ldots\ldots\ldots\text{(ii)}$$

Equation (i) may be used for determining T by measuring h by means of a travelling microscope. Equation (ii) may also be used for determining T if θ is already known. Or, if T is determined from (i), equation (ii) may be used for

finding θ. A travelling microscope or a spherometer may be used for measuring H.

In determining h some difficulty may be experienced in finding the exact position V, where the surface is vertical. This difficulty has been got over by Edser as follows: The objective of the microscope is provided with a plane glass plate P (Fig. 91), which both reflects and transmits light. Light from a source S is focussed by means of the converging lens L upon the rim of the drop after reflection at P. At the exact position of V a thin bright horizontal line will be seen. The micróscope is adjusted so that this line coincides with the horizintal cross-wire, and the vernier reading is noted. The microscope is then screwed (to move) up (and forwards) until the image of the *flat* top of the drop is focussed on the cross-wire. This observation is facilitated by thinly scattering some fine powder (such as lycopodium powder) on the plane top of the drop. The difference between this reading and the previous one gives h.

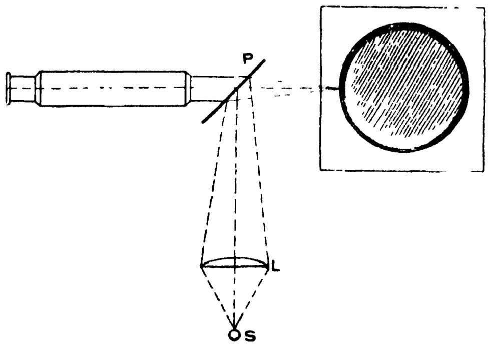

Fig. 91.

The above equations apply not only to a large *drop* but also to a large *bubble* of air under a glass plate, for example, in water. Figs. 88 and 90 must be turned upside down to represent the case of the bubble. Also θ here, being the angle inside the bubble, is the *supplement* of the angle of contact.

From equation (ii) it is clear that the thickness (H) of a large drop or bubble is independent of the volume of the drop or bubble.

Eliminating T from equations (i) and (ii), we get

$$\sin\frac{\theta}{2} = \frac{H}{h\sqrt{2}}$$

Hence θ can be found directly from H and h.

82. Variation of Surface Tension with Temperature.—(i) *Jaeger's Method.* In this method the pressure required to force a bubble of air, through a narrow orifice, into the liquid is found at various temperatures.

The apparatus is represented in Fig. 92. R is the

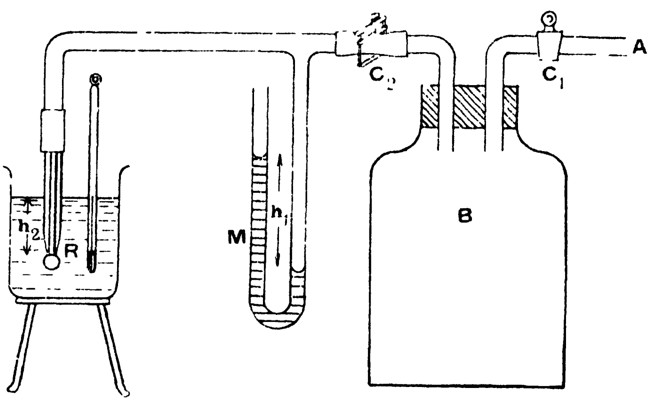

Fig. 92.

orifice of a tube dipped into the liquid whose surface tension is to be determined. The pressure of air required to force the bubble is found from the manometer M. B is a big bottle serving as the reservoir for compressed air.

With stop-cock C_2 closed and C_1 open, air is blown through A so as to store up, on closing C_1, as much compressed

air in B as possible. Now C_2 is gradually opened until bubbles of air are *slowly* liberated from R. The maximum difference of level h_1 indicated by M is then noted, and the depth h_2 of R below the liquid surface is measured. The radius r of the orifice R is measured with a travelling microscope.

Let T be the surface tension of the liquid and ρ_2 its density. To force a bubble through R the pressure required, above the atmospheric pressure, is $h_2 \rho_2 g + 2T / r$. This must be equal to $h_1 \rho_1 g$, the pressure indicated by the manometer (above the atmospheric pressure), where ρ_1 is the density of the liquid in the manometer. Hence

$$h_1 \rho_1 g = h_2 \rho_2 g + 2T / r.$$

$$\therefore \quad T = \tfrac{1}{2} (h_1 \rho_1 - h_2 \rho_2) \, g \, r.$$

This method is not very accurate for absolute values of T, but is very useful for comparison of surface tensions of various liquids or of the same liquid at various temperatures.

(ii) *U – Tube Method.* The U-tube here consists of a capillary tube for one limb and a tube of wide bore for the other limb. The tubes must be calibrated by the mercury thread method so that the radius at any required point may be known. The liquid, whose surface tension is required, is introduced into the tube so as to occupy convenient levels in the limbs. The tube is then immersed vertically in a suitable bath, so that, while the liquid levels in the tube are below the level of the bath, the open ends of the tube are well above that level (Fig. 93). A thermometer and stirrer (not shown in Fig.) are introduced into the bath.

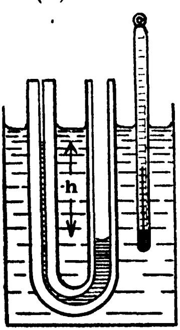

Fig. 93.

The bath is heated, and at each temperature at which the surface tension is required the difference in level h of the liquid in the two limbs is found

Let r_1 be the radius of the capillary bore and r_2 the radius of the wider bore, each at the corresponding meniscus. Let P be the atmospheric pressure, and let the liquid be one which wets glass, so that the angle of contact may be taken as zero. Then the pressures just below the concave surfaces of the liquid in the narrow and wide bores are $P - 2T/r_1$ and $P - 2T/r_2$ respectively. As the difference in level is h,

$(P - 2T/r_2) - (P - 2T/r_1) = h\rho g$, where ρ is the density of the liquid. Hence

$$T = h\rho g r_1 r_2 / 2(r_2 - r_1).$$

In the case of mercury, the meniscus in the narrow bore is *lower* than that in the wide bore. Let θ be the angle of contact. Then the pressures just below the convex surfaces in the narrow and wide bores are respectively

$$P + \frac{2T\cos(180^\circ - \theta)}{r_1} \quad \text{and} \quad P + \frac{2T\cos(180^\circ - \theta)}{r_2}$$

$$\therefore \frac{2T\cos(180^\circ - \theta)}{r_1} - \frac{2T\cos(180^\circ - \theta)}{r_2} = h\rho g. \text{ Hence}$$

$$T = h\rho g r_1 r_2 / 2(r_2 - r_1)\cos(180^\circ - \theta).$$

EXAMPLES

1. Calculate the work done on the film in blowing a soap-bubble from a diameter of 3 cm to one of 30 cm., if its surface tension be 45 c. g. s. units. [M.U., B.A.]

Increase of area $= 2 \times 4\pi \times 900/4 - 2 \times 4\pi \times 9/4$. sq. cm.

Mechanical work done in increasing the area = $T \times$ increase of area $= 45(2 \times 4\pi \times 900 / 4 - 2 \times 4\pi \times 9 / 4)$

$$= 2{\cdot}52 \times 10^5 \text{ ergs.}$$

2. A spherical drop of water of radius 1 mm. is sprayed into a million drops, all of the same size. Find the work expended in doing this. Surface tension of water = 74 dynes / cm.

Let r be the radius of each of the million drops. Then equating the volumes, $\frac{4}{3}\pi r^3 \times 10^6 = \frac{4}{3}\pi \times 0{\cdot}001$.

$\therefore \quad r^3 = 1 / 10^9$; hence $r = 1/10^3$ cm

Increase in area $= (4\pi / 10^6) \times 10^6 - 4\pi \times 0{\cdot}01$

$$= 4\pi (1 - 0{\cdot}01) \text{ sq. cm.}$$

$\therefore \quad$ Work expended $= 74 \times 4\pi (1 - 0{\cdot}01) = 920$ ergs.

3. A drop of water weighing 0·1 gm. is introduced between two plane and parallel glass plates. Calculate the force between the plates when they are at a distance of 0·0001 cm. apart. $T = 75$ dynes / cm.

Volume of drop = 0·1 c. c.

Area of *each* plate wetted = 0·1 / 0·0001 = 1000 sq. cm.

$F = 2\,AT / d$ (Art. 68) $= 2 \times 1000 \times 75 / 0{\cdot}0001 =$
$1{\cdot}5 \times 10^9$ dynes (*i. e.*, 1·5 tons wt. nearly).

4. Two soap bubbles of radii r_1 and r_2 blown from the same solution are allowed to coalesce into a single bubble of radius R. Prove that the tension of the bubble is to the atmospheric pressure as $R^3 - r_1^3 - r_2^3$ is to $4\,(r_1^2 + r_2^2 - R^2)$.

[M. U., B. A. & B. Sc. March 1927]

By Boyle's Law for mixture (Art 40, cor. 2),

$$\frac{P_1 V_1}{T_1} + \frac{P_2 V_2}{T_2} = \frac{PV}{T}.$$

Since the temperature is the same here,

$$P_1 V_1 + P_2 V_2 = P V, \ i.\ e.,$$

$$(p + 4\,T\,/\,r_1)\ \tfrac{4}{3}\,\pi\,r_1^3 + (p + 4\,T\,/\,r_2)\ \tfrac{4}{3}\,\pi\,r_2^3 =$$
$$(p + 4\,T\,/\,R)\ \tfrac{4}{3}\,\pi\,R^3,$$

where p is the atmospheric pressure and T the surface tension.

Hence $4\,T\,(r_1^2 + r_2^2 - R^2) = p\,(R^3 - r_1^3 - r_2^3)$, etc.

5. The difference of the levels of mercury in the two limbs of a U-tube is 0·88 cm. If the diameter of one limb is 1 mm. and that of the other 8 mm., calculate the surface tension of mercury. Density of mercury = 13·6 gm. per c. c. Angle of contact with walls of tube = 140°.

$$2\,T\cos\,(180^\circ - \theta)\,/\,r_1 - 2\,T\,(\cos 180^\circ - \theta)\,/\,r_2 = h\,\rho\,g$$
(Art. 82, ii).

$$i.\ e.,\ 2\,T\cos 40^\circ\,/\,0{\cdot}05 - 2\,T\cos 40^\circ\,/\,0{\cdot}4 =$$
$$0{\cdot}88 \times 13{\cdot}6 \times 981.$$

Hence T = 438 dynes per cm.

6. A rectangular glass plate of length 10·0 cm., breadth 2·54 cm., and thickness 2·00 mm., weighs 13·21 gm. in air. If it is held vertically, with its long edges horizontal and its lower half immersed in water, what will be its apparent weight? Surface tension of water = 72 dynes / cm.

Volume of plate immersed $= 10 \times 2{\cdot}54 \times 0{\cdot}2\,/\,2.$
$= 2{\cdot}54$ c.c.

$\therefore$ Force of buoyancy $= 2{\cdot}54$ gm. wt.

Pull due to surface tension downwards

$$= 2\,(10 + 0{\cdot}2)\ 72 \text{ dynes}$$
$$= 1{\cdot}50 \text{ gm. wt.}$$

$\therefore$ The apparent weight of the plate =

$$13{\cdot}21 - 2{\cdot}54 + 1{\cdot}50 = 12{\cdot}17 \text{ gm.}$$

7. A metal bar can stand a tension of 12,000 lb. wt. per sq. in. Find what pressure a hollow sphere of this material, of radius 4 ft. and thickness $\frac{1}{5}$ in., can stand without bursting.

Imagine a section of the sphere. A length of 1 ft. of the edge will have an area of $1 \times 1/5 \times 12$ sq ft. and hence will stand a tension of $12{,}000 \times 12 \times 12 \times 1/5 \times 12$ or 28,800 lb. wt. This corresponds to T in surface tension. Hence the pressure the sphere can stand (above the atmospheric pressure) $= 2\,T/r = 2 \times 28{,}800/4 = 14{,}400$ lb. wt. per sq. ft.

8. A cylindrical boiler is built with hemispherical ends. If the diameter of the cylinder is 6 ft., find the tensions in the different parts of the boiler when the pressure of the steam is 200 lb. wt. per sq. in. above the atmospheric pressure.

200 lb. wt. per sq. in. $= 200 \times 12 \times 12$ or

28.800 lb. wt. per sq. ft.

Let the tension developed in the material on account of this excess of internal pressure be T_1 (lb. wt. per ft.) in the cylindrical part and T_2 in the spherical part. Then

$$T_1 / 3 = 28{,}800 \text{ (Art. 66) and}$$
$$2\,T_2 / 3 = 28{,}800 \text{ (Art. 65).}$$
$$\therefore\ T_1 = 86{,}400 \text{ lb. wt. per ft. and}$$
$$T_2 = 43{,}200 \text{ lb. wt. per ft,}$$

9. A glass vessel with a flat top is filled with a liquid, except that a large flat air bubble remains under the central part of the top. This bubble is found to be 5·35 mm. deep over its central portion. The widest part of the bubble is 1·50 mm. below the under surface of the glass top. Assuming the density of the liquid is 1·00 gm. per c.c., calculate the surface tension and the angle of contact with the glass.

$$g = 978 \text{ cm.} / \text{sec.}^2$$

Derive the formulae you employ in the above calculation. [M.U., B. Sc., Sep. 1935]

Referring to Quincke's method of determining T and θ (Art. 81) we have here

$$H = 5{\cdot}35 \text{ mm.} = 0{\cdot}535 \text{ cm.}$$

$$h = 5{\cdot}35 - 1{\cdot}50 \text{ mm.} = 0{\cdot}385 \text{ cm.}$$

$$\rho = 1 \quad ; \quad g = 978.$$

Substituting these in the equation

$$T = \tfrac{1}{2} h^2 \rho g,$$

$$T = \tfrac{1}{2} \times 0{\cdot}385 \times 0{\cdot}385 \times 1 \times 978$$

$$= 72{\cdot}5 \text{ dynes} / \text{cm.}$$

Again, from the equation

$$\sin \tfrac{1}{2} \theta = H / h \sqrt{2}$$

or from the equation

$$T = H^2 \rho g / 2 (1 - \cos \theta),$$

$$\sin \tfrac{1}{2} \theta = 107 / 109 = 1 \text{ nearly.}$$

$\therefore \theta = 180^\circ$ nearly. This gives the angle within the bubble. The angle of contact between the liquid and glass is therefore nearly 0°.

10. (i) Water rises to a height of 4 cm. in a capillary tube. If the tube is depressed until only a length of 3 cm. is left above the water surface, what will happen ?

(ii) Into a vertical U-tube, consisting of a capillary tube 10 cm. long for one limb and a very wide tube 15 cm. long for the other, water is slowly introduced through the wide tube. What will be the difference of level and the curvature of the meniscus in the capillary tube at the various stages?

(i) According to the law of conservation of energy, water cannot go on flowing up, like a fountain, 1 cm. high above the tube. What happens here is that the curvature of the meniscus adjusts itself so that $2\,T\,/\,R$ becomes equal to $hg\rho$, where h is the height of the tube left above the liquid surface.

(ii) Let r be the radius of the capillary bore. Then ignoring the capillarity effect of the wide bore, the difference in level h will be given by $2\,T/r = h\,\rho\,g$ until the meniscus in the capillary bore (which is, of course, at a higher level now) reaches the top. On further introduction of water, the curvature $(1\,/\,R)$ of this meniscus so adjusts itself that $2\,T\,/\,R = h'\,\rho\,g$. When the two levels become equal, the surface of this meniscus becomes plane. When the level in the wide tube becomes higher, water does not, even now, flow out of the capillary tube, but its meniscus becomes *convex* to balance the extra pressure up to $2T/r$. After this, water overflows.

QUESTIONS ON CHAPTER VI

(1) Calculate the pressure inside a spherical cavity within a mass of water, if the cavity is 0·001 cm. in radius and at a depth of 10 cm. below the surface of the water. T = 78 c. g. s. units. Atm. pr. = 76 cm.

[M. U., B. A. & B. Sc., Sep. 1928]

(2) A drop of water 0·5 cm. radius is split into 1000 tiny drops. Find the increase in the surface energy.

Calculate the pressure inside one of these smaller drops. S. T. of water = 78 dynes / cm.

[M. U., B. A., & B. Sc., Sep. 1929]

(3) Two soap bubbles of radii r_1 and r_2 coalesce into one soap bubble of radius R. If P be the atmospheric pressure, calculate the surface tension. Hence show that the surface of the film shrinks and the contained air expands simultaneously. [M. U., B. Sc., March 1932]

(4) Calculate the work done in blowing a soap-bubble of radius 2 cm., with soap solution, the surface tension of which is 25 c. g. s. units. [M. U., B. A., March 1933]

(5) Show that in the case of a cylindrical boiler closed at both ends by spherical caps, the hemispherical ends need be only half as thick as the cylindrical portion.

[M. U., B. Sc., March 1933]

(6) Prove that the air inside a soap bubble of sufficiently large radius R would at atmospheric pressure P fill a sphere of radius $R + 4\,T\,/\,3\,P$, where T is the surface tension. [M. U., B. A., Sep. 1934]

(7) A glass capillary tube of internal diameter 0·05 cm. was clamped vertically with its lower end dipping 1·11 cm. below the surface of water in a beaker. It was found that, in order just to blow air out at the bottom of the tube, the air had to be at a pressure of 6940 dynes per sq. cm. in excess of the atmospheric. Deduce the value of the surface tension of the water-air surface. Density of water = 0·996 gm. / c. c. and accel. due to gravity = 978 cm. / sec.^{2}

[M. U., B. Sc., March 1935]

(8) Calculate the work done on the film in blowing a soap-bubble from a diameter of 3 cm., to one of 15 cm., if its surface tension be 30 C. G. S. units.

[M. U., B. A., March 1936]

(9) A boiler is built with a cylindrical body and hemispherical ends. If the common diameter be 5 ft., find the tension in the different parts of the boiler when the pressure of the steam inside is 200 lb. wt. per sq. in.

[M. U., B.A., Sep. 1936]

(10) A minute spherical bubble of air is rising slowly through a column of mercury in a tall jar. If the radius of the bubble at a depth of 100 cm. is 0·1 mm., at what depth below the surface will it have a radius equal to 0·12 mm. ?

Surface tension of mercury	=	520 dynes / cm.
Atmospheric pressure	=	76 cm., of mercury.
Density of mercury	=	13·6 gm. per c.c.
Accel. due to gravity	=	978 cm., per sec.^{2}

[M. U., B. Sc., Sep. 1936]

(11) Describe experiments to illustrate that the surface of a liquid is in a state of tension.

Prove that the surface tension of a liquid is the same in all directions and at all points.

The difference of pressure between the inside and outside of a soap bubble of diameter 6 mm. is 8 mm. of an oil of sp. gr. 0·7. Calculate the surface tension of the soap solution. [M. U., B.A., March 1937]

(12) Compare the surface of a liquid at rest with an elastic membrane under tension.

Find an expression for the excess of pressure inside a spherical soap bubble.

Calculate the depression of the mercury surface below the general level inside a vertical capillary tube of circular bore of 0·2 mm. diameter, with its lower end immersed in mercury.

Surface tension of mercury = 550 dynes / cm.

Angle of contact of mercury = 130°.

Density of mercury = 13·6 gm. per c.c.

[M. U., B. Sc., Sep. 1937]

(13) Find the force between two parallel plates held together by a liquid film.

A drop of water weighing 0·1 gm. is introduced between two plane and parallel metal plates. What force will be exerted when the plates are at a distance of 0·0001 cm. apart. [M. U., B. Sc., March 1938]

(14) Define 'surface tension' of a liquid and show that it is equal to the (mechanical) energy per unit area of the surface.

Calculate the work done on the film in blowing a soap-bubble from a diameter of 3 cm. to one of 30 cm. if its surface tension be 45 in c. g. s. units.

[M. U., B.A., March 1938]

(15) Find an expression for the excess of pressure inside a spherical soap-bubble.

The limbs of a U-tube are vertical and have internal diameters of 5 and 1 mm. respectively. If the tube contains water, what will be the difference in the surface levels in the limbs? [M. U., B.A., Sep. 1938]

(16) Find an expression for the excess of pressure inside a soap-bubble over that outside.

Two bubbles of radii r_1 and r_2 coalesce into one bubble. Assuming the temperature to remain constant, find the radius of the joint bubble and also the pressure inside the bubble. [M. U., B. A., March 1939]

(17) Explain why water spreads on a clean mercury surface, but collects into a drop when the surface is greasy.

Describe how the angle of contact of mercury with glass may be determined experimentally.

[M. U., B. A., Sep. 1939]

(18) Define 'surface energy,' 'angle of contact'.

Describe a method of determining the angle of contact between mercury and glass.

Find the condition for the formation of a drop of one liquid on the surface of another.

[M. U., B. Sc., Sep. 1939]

(19) Define 'angle of contact'. Describe how the angle of contact of mercury with glass may be experimentally determined.

A capillary tube 0·1 mm. in diameter is dipped vertically in water. Find the height to which water rises in the tube. The surface tension of water is 74 dynes per cm.

[M. U., B.A., March 1940]

APPENDIX

POINTS TO BE REMEMBERED

1 c. ft. of water weighs 62·4 lb. (roughly, 62·5 lb.) at 4°C.

Density of air $= 0·001293$ gm. per. c.c. at N.T.P.

Density of hydrogen $= 0·0000899$,, ,,

Density of mercury $= 13·596$ gm. per. c.c. at 0°C., and

13·522 ,, at 30°C.

Density of water at 29°C. $= 0·996$ gm. per. c.c.

Coefft. of exp. of mercury $= 0·000182$ per deg. C.

Coefft. of lin. exp. of brass $= 0·000019$,,

Accel. due to gravity $= 978$ cm./sec.^{2} at Madras and 980·6 cm./sec.^{2} at sea-level at latitude 45°;

$= 32$ ft./sec.^{2} (roughly).

Normal atmospheric pressure or standard atmospheric pressure $= 1033$ gm. wt. per sq. cm. or $1·013 \times 10^6$ dynes per sq. cm. or 14·7 lb. wt. per sq. in.

Height of homogeneous atmosphere $= 7990$ metres or 26,280 ft.

Sat. vap. pressure of water at 29°C. $= 30$ mm. of merc.

Sat. vap. pressure of mercury at 20°C. $= 0·001$ mm. of merc.

,, ,, 50°C. $= 0·01$,,

Surface tension of water at 30°C. $= 70$ dynes/cm. (roughly).

Surface tension of mercury at 0°C. = 550 dynes/cm.
(roughly).

Angle of contact between mercury and glass in air
= 140° nearly.

$\pi = 3{\cdot}1416$	$\log_e 10 = 2{\cdot}3026$
$\log \pi = 0{\cdot}4971$	$\log 2{\cdot}3026 = 0{\cdot}3622$
$\log 2 = 0{\cdot}3010$	$\log 273 = 2{\cdot}4362$
$\log 3 = 0{\cdot}4771$	$\log 760 = 2{\cdot}8808$
$\log 4 = 0{\cdot}6021$	$\log 978 = 2{\cdot}9903$
$\log 5 = 0{\cdot}6991$	$\log 0{\cdot}001293 = \bar{3}{\cdot}1116$

The above logarithms are to the base 10 where not otherwise indicated.

ANSWERS

CHAPTER I

(1) If b is the altitude of the rectangle, first and second lines are to be $b/\sqrt{3}$ and $b\sqrt{2}/\sqrt{3}$ below top side.

(2) DE $= \frac{3}{4}$ DC, where E is the point at which the line cuts DC.

(3) Line bisecting other two sides.

(4) $\frac{1}{2}\, ag\, [b\,(2a-b)\,\rho_1 + (a-b)^2\,\rho_2]$

(5) Equal to the weight of the liquid and acting vertically along the axis.

(9) $1{\cdot}297 \times 10^6$ lb. wt.; 30·38 ft.

(10) Line to be drawn $b/\sqrt[4]{2}$ vertically below vertex, where b is the altitude of the triangle.

(11) 8967 lb. wt.

(12) & **(13)** See example 1.

CHAPTER II

(1) 20·9 tons wt.

(5) 30,326 lb. wt. at $4\frac{2}{3}$ ft. above bottom of gate on the central verticle line.

(6) 16,875 lb. wt.; $3\frac{4}{5}$ ft. above bottom of gate on the central vertical line.

(7) See example 3.

(8) 23,676 lb. wt.; 4·11 ft. above bottom of gate on the central vertical line.

Hint. Thrust on first side $= 5 \times 62\frac{1}{2} \times 12 \times 10$ lb. wt.
$= F_1$, say.

Thrust on second side $= 3 \times 64 \times 12 \times 6$ lb. wt.
$= F_2$, say.

Resultant thrust $= F_1 - F_2$.

Let the point of action be distant y above the bottom of the gate. Then

$$(F_1 - F_2)\, y = F_1 \times 10/3 - F_2 \times 6/3.$$

(9) 40,000 lb. wt.; $4\frac{1}{3}$ ft. above bottom of gate on the central vertical line.

CHAPTER III

(1) $r / \sqrt{2}$ above hemispherical part.

(2) Let A be the area of cross-section and x and y the depths of immersion. Then

$$h\, s\, A = A\, x \times 1 + A\, (h - x)\, d \quad \text{and}$$

$$h\, s\, A = A\, y \times 1. \quad \text{Hence } y - x = \text{etc.}$$

(4) 10^4 c. ft. [*Hint.* $x \times 8 / 100 - x \times 8 \times 0{\cdot}07 / 100 = 750.$]

(5) $6\, s\, (1 - s) < 1$

(8) 1 : 1·015.

(9) $r > h / \sqrt{2}$.

(10) 4 ft.

(13) 5 in.

(15) 1·16.

CHAPTER IV

(1) 132 ft.

(2) 5·06 ft. [*Hint.* $2{\cdot}5 \times 13{\cdot}6 \times 8 =$

$$6\left[\, (x + 6)\, 1{\cdot}025 + 2{\cdot}5 \times 13{\cdot}6\right].$$

(3) $(h_1 - h_2)\, b\, /\, (a - b)$ and $(h_1 - h_2)\, a\, /\, (a - b)$.

[*Hint.* The pressures of air above the mercury in the two cases are a and b. Let x be the length of the col. of air in the first case. Then the length of the col. in the second case is $x + h_1 - h_2$. Applying Boyle's law, $x\, a = (x + h_1 - h_2)\, b$, etc.]

(5) 0·84 kilometre. Mean temp. (which must be given in the problem) is taken to be 0°C.

(7) *Hint.* Press. of air above mercury = 0·5 and 0·75 in. resp. in the first and second cases. Let l be the length of air col. above mercury in the first case. Then the length in the second case $= l - 1$. By Boyle's law, $l \times 0{\cdot}5 = (l - 1)\, 0{\cdot}75$. Hence $l = 3$ in. $\therefore$ Length of tube above mercury in cistern $= 28 + 3 = 31$ in. $\therefore$ With merc. col. x, length of air col. $= 31 - x$. Its pressure $= 3 \times 0{\cdot}5\, /\, (31 - x) = 3\, /\, (62 - 2\,x)$. Hence press. of atm. $= x + 3\, /\, (62 - 2\,x)$.

(8) 495 metres.

(9) 0·92 kilometre.

(10) 1·33 kilometres, taking the mean temp. to be 0°C.

[The necessary data that can be taken from the tables do not include the mean temperature between the places].

CHAPTER VI

(1) $1{\cdot}18 \times 10^6$ dynes per sq. cm.

[*Hint.* $P = 76 \times 13{\cdot}6 \times 981 + 10 \times 981 + 2 \times 78 / 0{\cdot}001$.]

(2) Mechanical work expended (not increase in the surface energy, see Art. 62) = 2205 ergs.

Pressure inside = 3120 dynes / sq. cm. in excess of external pressure.

(3) See example 4.

(4) 2513 ergs.

(5) The pressure inside is greater than the pressure outside by T_1 / R in the case of the cylinder and $2\,T_2 / R$ in the case of the sphere (see example 8). Since the pressure inside is the same for both, $2\,T_2 / R = T_1 / R$: hence $T_2 = T_1 / 2$, etc.

(6) Pressure of air inside soap bubble $= P + 4\,T / R$.

Its vol. $= 4\,\pi\,R^3 / 3$. Let x be the radius of the sphere occupied by the air at pressure P. Then by Boyle's law,

$(P + 4\,T / R) \times 4\,\pi\,R^3 / 3 = P \times 4\,\pi\,x^3 / 3$. Hence

$x^3 = (P + 4\,T/R)\,R^3 / P = (1 + 4\,T/RP)\,R^3$.

$\therefore\ x = R\,(1 + 4T/RP)^{\frac{1}{3}} = R\,(1 + \frac{1}{3}\,4T/RP)$, ignoring higher powers of $4T/RP$, since R is given to be large.

(7) $73{\cdot}2$ dynes per cm.

[*Hint.* $6940 = 1{\cdot}11 \times 0{\cdot}996 \times 978 + 2\,T / 0{\cdot}025$.]

(8) 20,360 ergs.

(9) Tensions in cylindrical and hemispherical parts are 72,000 lb. wt. per ft. and 36,000 lb. wt. per ft. respectively. Pressure of steam is taken to be 200 lb. wt. per sq. in. *above the atmos. pressure.*

(10) 24·5 cm. [*Hint.* Apply Boyle's law, $p_1v_1 = p_2v_2$.
$p_1 = 176 \times 13{\cdot}6 \times 978 + 2 \times 520/0{\cdot}01$;
$v_1 = \frac{4}{3}\pi \times 0{\cdot}01^3$.
$p_2 = (x + 76)\, 13{\cdot}6 \times 978 + 2 \times 520 / 0{\cdot}012$;
$v_2 = \frac{4}{3}\pi \times 0{\cdot}012^3$.]

(11) 41 dynes per cm. [*Hint.* $4T/0{\cdot}3 = 0{\cdot}8 \times 0{\cdot}7 \times 980$.]

(12) 5·3 cm.

(13) 15×10^8 dynes, taking T as 75 dynes/cm.

(14) $25{\cdot}2 \times 10^4$ ergs.

(15) 2·45 cm., taking T as 75 dynes/cm.

(16) Refer to example 4.

(19) 30·2 cm.

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